# Orthogonal-polynomial ensembles from the reciprocal Riemann $\xi$-function

Kenan

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**Abstract**

Let

$$\mathcal{X}(x) = \xi\left(\frac{1}{2} + x\right), \qquad \mathrm{d}\mu(x) = Z^{-1}\mathcal{X}(x)^{-1}\,\mathrm{d}x,$$

and let $b_n$ be the off-diagonal recurrence coefficients of the orthonormal polynomials for $\mu$. We develop the associated orthogonal-polynomial ensemble and its confluent Hankel determinants. First, a GHS inequality for the classical Riemann density, combined with an exact virial identity and a concavity argument, gives the all-index bounds

$$b_n \geq \frac{n}{Q'(a_n)} = \frac{na_n}{2n-1}, \qquad a_n Q'(a_n) = 2n - 1,$$

and $b_n^2 \geq b_1^2$, where $Q = \log \mathcal{X}$. Second, a Mhaskar–Rakhmanov–Saff analysis and a quantitative complex log-Freud recurrence theorem yield

$$b_n = \frac{\pi(n - 7/8)}{W_0(2(n-7/8)/e)} + O\left(\frac{\log n}{n}\right).$$

The $7/8$ shift is retained by an order-$1/n$ cancellation in the $\rho = 0$ Riemann–Hilbert recurrence functional. Consequences include an additive Lambert-$W$ law, eventual monotonicity, a Breuer–Duits central limit theorem for every polynomial statistic, the exact identity $\mathrm{Var}(\sum_j X_j) = b_n^2$, an explicit Ullman law for both particles and polynomial zeros, and the free energy

$$\log H_n(0) = n^2\left(\log n - \log\log n + \log\pi - \frac{3}{2}\right) + o(n^2).$$

All statements are unconditional and concern the positive weight on the real axis. Explicit premise-matched entire functions with zeros off the imaginary axis show that these results do not imply the Riemann hypothesis. A pinned interval-arithmetic and symbolic verification artifact accompanies the paper.

**Keywords.** Riemann xi-function; orthogonal polynomials; unitary ensemble; Mhaskar–Rakhmanov–Saff numbers; Lambert $W$; linear statistics.

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# 1 Introduction

The completed Riemann zeta-function

$$\xi(s)=\frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$$

is entire and satisfies $\xi(s)=\xi(1-s)$. Throughout the paper we use the real-centered convention

$$\mathcal{X}(z)=\xi\left(\frac{1}{2}+z\right),$$

so that the Riemann hypothesis is the assertion that every zero of $\mathcal{X}$ lies on the imaginary axis. The function $\mathcal{X}$ is even, strictly positive on $\mathbb{R}$, and grows sufficiently rapidly that $\mathcal{X}^{-1}$ has moments of every order. We therefore obtain a canonical probability measure

$$\mathrm{d}\mu(x)=Z^{-1}e^{-Q(x)}\,\mathrm{d}x,\qquad Q(x)=\log\mathcal{X}(x),\qquad Z=\int_{\mathbb{R}}\frac{\mathrm{d}x}{\mathcal{X}(x)}.\tag{1}$$

The object of this paper is the orthogonal-polynomial and random-matrix theory generated by (1). This is different from the classical use of the Fourier kernel whose transform is $\xi$: here the completed zeta-function itself is the denominator of the spectral weight. The reciprocal transform has appeared naturally in Schoenberg-type formulations of the Riemann hypothesis [3]. Our emphasis is instead on unconditional real-axis structure.

Let $p_{-1}=0, p_0=1$ and let $p_n$ be the orthonormal polynomials for $\mu$. Evenness removes the diagonal recurrence coefficients:

$$xp_n(x)=b_{n+1}p_{n+1}(x)+b_np_{n-1}(x),\qquad b_n>0,\quad b_0=0.\tag{2}$$

The main theorem summarizes the resulting asymptotic picture.

**Theorem 1.1** (Reciprocal-$\xi$ ensemble)**.** *Let $b_n$ be defined by (2), let $W_0$ denote the principal Lambert function, and put $W_n=W_0(2n/e)$. Then:*

(i) *If $a_nQ'(a_n)=2n-1$, then*

$$b_n\geq\frac{n}{Q'(a_n)}=\frac{na_n}{2n-1},\qquad b_n>\frac{2n}{\log(4n-2)}\quad(n\geq 2),$$

*and $b_n^2\geq b_1^2$ for every $n\geq 1$.*

(ii) *The recurrence coefficients satisfy*

$$b_n=\frac{\pi(n-7/8)}{W_0(2(n-7/8)/e)}+O\left(\frac{\log n}{n}\right).\tag{3}$$

*Consequently*

$$\frac{n}{W_n}\left(\frac{\pi n}{b_n}-W_n\right)\longrightarrow\frac{7}{8},\quad n\left(\frac{b_{n+1}}{b_n}-1\right)\longrightarrow 1,\quad (1+W_n)(b_{n+1}-b_n)\longrightarrow\pi.$$

*In particular, $b_n$ is eventually strictly increasing.*

(iii) *For the $n$-particle orthogonal-polynomial ensemble associated with $\mu$, every centered polynomial statistic at scale $b_n$ converges to the Gaussian law determined by the Laurent symbol $z+z^{-1}$. For the trace $S_n=\sum_{j=1}^n X_j$,*

$$\mathbb{E}S_n=0, \qquad \mathrm{Var}(S_n)=b_n^2, \qquad \frac{S_n}{b_n}\Longrightarrow N(0,1).$$

(iv) *Both the empirical particle measure and the zero-counting measure of $p_n$, after division by $b_n$, converge to the probability law*

$$\mathrm{d}\nu_{\mathrm{U}}(x)=\frac{1}{2\pi}\operatorname{arcosh}\left(\frac{2}{|x|}\right)\mathbf{1}_{\{0<|x|<2\}}\,\mathrm{d}x.$$

*Particle convergence is weak in probability and zero convergence is weak deterministic convergence.*

(v) *If*

$$F(t)=\int_{\mathbb{R}}\frac{e^{itx}}{\mathcal{X}(x)}\,\mathrm{d}x, \qquad H_n(t)=(-1)^{n(n-1)/2}\det\Big[F^{(j+k)}(t)\Big]_{j,k=0}^{n-1},$$

*then*

$$\log H_n(0)=n^2\left(\log n-\log\log n+\log\pi-\frac{3}{2}\right)+o(n^2),$$

*and, uniformly for $u$ in compact subsets of $\mathbb{R}$,*

$$\frac{H_n(u/b_n)}{H_n(0)}\longrightarrow e^{-u^2/2}.$$

The theorem combines several general theories, but the constants and their compatibility are specific. The coefficient $\pi$ comes from the completed-zeta gamma factor, the shift $7/8$ comes from the $7/(4x)$ term in the logarithmic derivative of $\mathcal{X}$, and the Ullman density arises from regular variation of index one. At finite index, the GHS inequality controls $Q'''$ and creates a useful concavity in the virial variable $xQ'(x)$.

The division between imported and new input is as follows. The qualitative recurrence theorem, complex log-Freud steepest descent, polynomial-statistic CLT, and regularly varying density theorem are general results cited below. The new deductions are the concavity mechanism in the virial variable, the all-index reciprocal-$\xi$ bounds, verification of the complex hypotheses and normalization, the source-level $\rho=0$ cancellation retaining the $7/8$ shift, and the resulting trace, Ullman, and free-energy specializations. Only the two rational inequalities in (19) require interval arithmetic.

No assertion in theorem 1.1 assumes the Riemann hypothesis. Nor can the conclusions be reversed to prove it. In section 9 we give three explicit even entire functions, positive on $\mathbb{R}$, which preserve the relevant real-axis mechanisms while having zeros off the imaginary axis. This logical separation is important: the paper concerns the orthogonal ensemble of the reciprocal weight, not the location of the complex zero divisor.

The paper is organized as follows. Section 2 records the exact Hankel, ensemble, Jacobi-compression, and Toda identities. Section 3 proves the all-index bounds. Section 4 derives the MRS expansion and section 5 extracts the quantitative recurrence asymptotics. The fluctuation and global laws are proved in sections 6 and 7; the free energy is computed in section 8. Section 9 discusses logical scope, and section 10 describes the reproducibility artifact.

## 2 Exact ensemble and determinant identities

All identities in this section hold for any positive even weight with all moments. We state them in the reciprocal-$\xi$ normalization to fix every factor and sign.

**Proposition 2.1** (Andreief and compression)**.** *For every $n \geq 1$,*

$$H_n(t) = \frac{1}{n!} \int_{\mathbb{R}^n} e^{it \sum_{\ell=1}^n x_\ell} \prod_{\ell=1}^n \frac{\mathrm{d}x_\ell}{\mathcal{X}(x_\ell)} \prod_{1 \leq j < k \leq n} (x_k - x_j)^2. \tag{4}$$

*In particular, $H_n(0) > 0$, and*

$$\frac{H_n(t)}{H_n(0)} = \mathbb{E} e^{itS_n}, \qquad S_n = \sum_{\ell=1}^n X_\ell, \tag{5}$$

*where $X_1, \ldots, X_n$ have the corresponding orthogonal-polynomial ensemble. If $J$ is the Jacobi matrix of $\mu$ and $P_n$ projects onto the first $n$ coordinate vectors, then*

$$\frac{H_n(t)}{H_n(0)} = \det\Big( P_n e^{itJ} P_n|_{\operatorname{ran} P_n} \Big). \tag{6}$$

*Proof.* Since

$$F^{(j+k)}(t) = i^{j+k} \int_{\mathbb{R}} x^{j+k} e^{itx} \mathcal{X}(x)^{-1} \, \mathrm{d}x,$$

factoring $i^j$ from row $j$ and $i^k$ from column $k$ produces $i^{n(n-1)} = (-1)^{n(n-1)/2}$, which cancels the sign in the definition of $H_n$. Andreief's identity then gives (4). Dividing by its value at $t = 0$ gives (5).

Changing from monomials to the orthonormal basis $p_0, \ldots, p_{n-1}$ leaves the determinant ratio unchanged and gives the matrix

$$\left[ \int_{\mathbb{R}} e^{itx} p_j(x) p_k(x) \, \mathrm{d}\mu(x) \right]_{j,k<n}.$$

Functional calculus for the multiplication operator identifies this matrix with the compression of $e^{itJ}$, proving (6). □

**Proposition 2.2** (Norm product and Toda identity)**.** *For $n \geq 1$,*

$$H_n(0) = Z^n \prod_{j=1}^{n-1} b_j^{2(n-j)}. \tag{7}$$

*Writing $D_n(t) = H_n(t)/H_n(0)$, with $D_0 = 1$, one has wherever the quotients are defined*

$$-(\log D_n(t))'' = b_n^2 \frac{D_{n-1}(t) D_{n+1}(t)}{D_n(t)^2}. \tag{8}$$

*Proof.* Let $h_j$ be the squared norm of the monic polynomial of degree $j$ for $\mu$. The moment Gram determinant is $\prod_{j=0}^{n-1} h_j$, while $h_0 = 1$ and $b_j^2 = h_j/h_{j-1}$. Restoring the unnormalized measure contributes $Z^n$, and telescoping gives (7). The contiguous Hankel determinant identity gives

$$-(\log H_n)'' = \frac{H_{n-1} H_{n+1}}{H_n^2}.$$

At $t = 0$, (7) makes the right-hand ratio equal to $b_n^2$. Dividing out the zero-time values proves (8). □

**Remark 2.3.** *The exact representations do not imply that $H_n(t)$ is positive for fixed $t \neq 0$. They identify the missing sign question with a trace characteristic function, a leading orthogonal-flow minor, or a Toda ratio, but none of these descriptions is sign-preserving by itself.*

## 3 A GHS–Jensen bound and the global floor

We first isolate a general inequality. Its hypotheses are designed to turn the third-derivative information supplied by the GHS theorem into a recurrence bound.

**Theorem 3.1** (GHS–Jensen recurrence bound)**.** *Let $Q \in C^3(\mathbb{R})$ be even, suppose $e^{-Q}$ has all moments and boundary terms vanish under polynomial integration by parts, and assume*

$$Q''(x) > 0, \qquad Q'''(x) \leq 0 \quad (x > 0).$$

*Let $b_n$ be the off-diagonal recurrence coefficients for the normalized weight $e^{-Q(x)}\,\mathrm{d}x$. For the unique $a_n > 0$ satisfying*

$$a_n Q'(a_n) = 2n - 1, \tag{9}$$

*one has*

$$b_n \geq \frac{n}{Q'(a_n)} = \frac{n a_n}{2n-1}. \tag{10}$$

*If $C = Q''(0)$, then also*

$$b_n^2 \geq \frac{n^2}{C(2n-1)}. \tag{11}$$

*Proof.* Put $f = Q'$, $g = Q''$, and $h = Q'''$ on $(0, \infty)$. Since $g$ is nonincreasing,

$$xg(x) \leq \int_0^x g(t)\,\mathrm{d}t = f(x).$$

Define $u(x) = xf(x)$, $v(x) = f(x)^2$, and $\phi(u(x)) = v(x)$. The function $u$ is strictly increasing. Direct differentiation gives the identity

$$\frac{(f + xg)^2}{2} \frac{\mathrm{d}}{\mathrm{d}x} \phi'(u(x)) = g^2(xg - f) + f^2 h \leq 0. \tag{12}$$

Thus $\phi$ is concave. By evenness, $\phi(xQ'(x)) = Q'(x)^2$ on all of $\mathbb{R}$.

The coefficient of $p_{n-1}$ in $p_n'$ is $n/b_n$. Integration by parts and orthogonality yield

$$\frac{n}{b_n} = \int_{\mathbb{R}} Q'(x) p_n(x) p_{n-1}(x) e^{-Q(x)} \frac{\mathrm{d}x}{\int_{\mathbb{R}} e^{-Q}}. \tag{13}$$

A second integration by parts, applied to $xp_{n-1}(x)^2 e^{-Q(x)}$, gives the exact virial identity

$$\int_{\mathbb{R}} xQ'(x) p_{n-1}(x)^2 \,\mathrm{d}\mu(x) = 2n - 1. \tag{14}$$

Indeed, the two polynomial terms contribute $1 + 2(n-1)$.

Cauchy–Schwarz in (13), Jensen's inequality for the probability measure $p_{n-1}^2\,\mathrm{d}\mu$, and (14) now give

$$\frac{n^2}{b_n^2} \leq \int Q'(x)^2 p_{n-1}(x)^2 \,\mathrm{d}\mu(x) = \int \phi(xQ'(x)) p_{n-1}(x)^2 \,\mathrm{d}\mu(x) \leq \phi(2n-1) = Q'(a_n)^2.$$

This proves (10).

Finally, $0 < g(x) \le C$ for $x > 0$, so

$$Q'(x)^2 \le CxQ'(x) \qquad (x \in \mathbb{R}).$$

Combining this estimate with (13) and (14) proves (11). $\square$

For the reciprocal-$\xi$ potential, the hypotheses come from the classical Riemann density. Newman proved that the derivative of its potential is convex [8]; the one-site theorem of Ellis–Newman [2] then gives a GHS inequality for its logarithmic Laplace transform.

**Lemma 3.2** (GHS curvature for $\mathcal{X}$). *For $Q(x) = \log \mathcal{X}(x)$,*

$$Q''(x) > 0 \quad (x \in \mathbb{R}), \qquad Q'''(x) \le 0 \quad (x > 0).$$

*Proof.* In Newman's normalization there is an even potential $V$ such that

$$\mathcal{X}(z) = 4 \int_{\mathbb{R}} e^{2zt - V(t)} \, \mathrm{d}t.$$

The second derivative of its logarithm is four times the variance under the exponentially tilted probability law, hence is strictly positive. Newman's convexity theorem for $V'$, together with the one-site GHS theorem of Ellis–Newman, says that the third derivative of this logarithmic Laplace transform is nonpositive for positive tilt. Rescaling the tilt by 2 preserves the sign. $\square$

We next make equation (10) explicit. The logarithmic derivative of the completed zeta-function is

$$Q'(x) = \frac{1}{s} + \frac{1}{s-1} - \frac{1}{2}\log \pi + \frac{1}{2}\psi(s/2) + \frac{\zeta'(s)}{\zeta(s)}, \qquad s = x + \frac{1}{2}. \tag{15}$$

**Lemma 3.3** (Elementary logarithmic-derivative bound). *For $x \ge 3$,*

$$Q'(x) \le \frac{1}{2}\log x. \tag{16}$$

*Proof.* For $s > 1$, $\zeta'(s)/\zeta(s) < 0$, and the standard digamma bound $\psi(y) \le \log y - 1/(2y)$ gives

$$Q'(x) \le \frac{1}{2}\log\frac{s}{2\pi} + \frac{1}{2s} + \frac{1}{s-1}.$$

The difference between $\frac{1}{2}\log x$ and the right-hand side is

$$R(x) = \frac{1}{2}\log\frac{2\pi x}{s} - \frac{1}{2s} - \frac{1}{s-1}.$$

It is strictly increasing because

$$R'(x) = \frac{1}{2}\left(\frac{1}{x} - \frac{1}{s}\right) + \frac{1}{2s^2} + \frac{1}{(s-1)^2} > 0.$$

Moreover,

$$R(3) = \frac{1}{2}\log\frac{12\pi}{7} - \frac{19}{35} > 0.$$

For example, the last inequality follows from $\pi > 3$ and $\log 3 > 1 + 1/12 + 1/80 = 38/35 + 17/1680$. $\square$

**Theorem 3.4** (All-index reciprocal-$\xi$ bounds)**.** *Let $a_n$ be defined by (9). Then*

$$b_n \geq \frac{n}{Q'(a_n)} = \frac{na_n}{2n-1},$$

*and, for $n \geq 2$,*

$$b_n > \frac{2n}{\log(4n-2)}. \tag{17}$$

*Furthermore,*

$$b_n^2 \geq b_1^2 \qquad (n \geq 1). \tag{18}$$

*Proof.* The first assertion follows from theorem 3.1 and lemma 3.2. Since $Q'(3) < 1$, the equation $a_n Q'(a_n) = 2n-1$ implies $a_n > 3$ for $n \geq 2$. By lemma 3.3,

$$a_n \log a_n \geq 4n-2.$$

If $N = 4n-2$, then

$$\frac{N}{\log N} \log \frac{N}{\log N} < N.$$

Monotonicity of $x \log x$ gives $a_n > N/\log N$, which inserted into (10) proves (17).

For the floor, the interval certificate described in section 10 proves

$$Q''(0) < \frac{1}{20}, \qquad b_1^2 < 24. \tag{19}$$

For $n \geq 2$,

$$\frac{n^2}{2n-1} \geq \frac{4}{3}$$

and hence (11) and (19) give

$$b_n^2 \geq \frac{4}{3Q''(0)} > b_1^2.$$

The case $n = 1$ is equality. □

**Remark 3.5.** *The lower bound has the correct order but not the sharp constant. In corollary 4.4 we show that its ratio to $b_n$ tends to $2/\pi$.*

# 4 The MRS scale and its $7/8$ shift

Let $R_n > 0$ be the Mhaskar–Rakhmanov–Saff radius for the square-root weight $e^{-Q/2}$, normalized by

$$n = \frac{1}{\pi} \int_0^1 \frac{R_n t\, Q'(R_n t)}{\sqrt{1-t^2}}\, \mathrm{d}t. \tag{20}$$

This is the normalization for which the recurrence coefficient is asymptotic to $R_n/2$.

**Lemma 4.1** (Completed-zeta expansion)**.** *As $x \to +\infty$,*

$$Q'(x) = \frac{1}{2} \log \frac{x}{2\pi} + \frac{7}{4x} + \frac{1}{48x^2} + O(x^{-3}), \tag{21}$$

$$Q''(x) = \frac{1}{2x} + O(x^{-2}), \qquad Q'''(x) = -\frac{1}{2x^2} + O(x^{-3}). \tag{22}$$

*Proof.* Insert the Stirling expansion of $\psi(s/2)$ into (15). The Dirichlet series for $\zeta'/\zeta(s)$ is exponentially small as $s \to +\infty$. Expanding $s = x + 1/2$ in powers of $x^{-1}$ gives respectively the coefficients 7/4 and 1/48. Differentiation gives (22). $\square$

**Proposition 4.2** (MRS expansion)**.** *The radius in* (20) *satisfies*

$$n = \frac{R_n}{2\pi}\left(\log\frac{R_n}{\pi} - 1\right) + \frac{7}{8} + O\left(\frac{\log R_n}{R_n}\right). \tag{23}$$

*Consequently,*

$$\frac{R_n}{2} = \frac{\pi(n - 7/8)}{W_0(2(n-7/8)/e)} + O\left(\frac{\log n}{n}\right). \tag{24}$$

*Proof.* The two exact integrals

$$\int_0^1 \frac{t}{\sqrt{1-t^2}}\,\mathrm{d}t = 1, \qquad \int_0^1 \frac{t\log t}{\sqrt{1-t^2}}\,\mathrm{d}t = \log 2 - 1 \tag{25}$$

give, from the leading term of (21),

$$\frac{R_n}{2\pi}\left(\log\frac{R_n}{2\pi} + \log 2 - 1\right) = \frac{R_n}{2\pi}\left(\log\frac{R_n}{\pi} - 1\right).$$

The $7/(4x)$ term contributes

$$\frac{1}{\pi}\frac{7}{4}\int_0^1 \frac{\mathrm{d}t}{\sqrt{1-t^2}} = \frac{7}{8}.$$

To control the remaining terms, split the integral at $t = A/R_n$ for a fixed sufficiently large $A > 0$. The small interval contributes $O(R_n^{-1})$ by analyticity and oddness of $Q'$ at the origin. On the complementary interval, (21) is uniform; its $x^{-2}$ term contributes $O(\log R_n/R_n)$, and the remainder is smaller. This proves (23).

Put $m = n - 7/8$ and

$$\mathcal{H}(R) = \frac{R}{2\pi}\left(\log\frac{R}{\pi} - 1\right).$$

The leading part of (23) first gives $R_n \asymp n/\log n$. The exact inverse of $\mathcal{H}(R) = m$ is

$$R = \frac{2\pi m}{W_0(2m/e)}.$$

Since $\mathcal{H}'(R) = (2\pi)^{-1}\log(R/\pi)$, the error in (23) changes the inverse by $O(\log n/n)$, proving (24). $\square$

The qualitative Freud theorem already determines the leading scale.

**Proposition 4.3** (Freud asymptotic)**.** *One has*

$$\frac{b_n}{R_n} \longrightarrow \frac{1}{2}, \tag{26}$$

*and therefore*

$$b_n \sim \frac{\pi n}{W_0(2n/e)}, \qquad \frac{b_n \log n}{n} \longrightarrow \pi. \tag{27}$$

*Proof.* Set $\mathcal{Q} = Q/2$. By lemmas 3.2 and 4.1, $e^{-\mathcal{Q}}$ is a very smooth Freud weight of order one in the sense of Lubinsky–Mhaskar–Saff [5, Definition 2.1]. Indeed,

$$\frac{x^2|\mathcal{Q}'''(x)|}{\mathcal{Q}'(x)} = O\left(\frac{1}{\log x}\right), \qquad 1 + \frac{x\mathcal{Q}''(x)}{\mathcal{Q}'(x)} \longrightarrow 1.$$

Their recurrence theorem [5, Theorem 2.3] gives (26). Combining it with (24) proves (27). □

**Corollary 4.4** (Sharpness of the virial scale)**.** *If $a_n Q'(a_n) = 2n-1$ and $L_n = n/Q'(a_n)$, then*

$$\frac{R_n}{a_n} \longrightarrow \frac{\pi}{2}, \qquad \frac{b_n}{a_n} \longrightarrow \frac{\pi}{4}, \qquad \frac{L_n}{b_n} \longrightarrow \frac{2}{\pi}.$$

*Proof.* Apply (21) to the defining equations for $a_n$ and $R_n$, then use (26) and $L_n = na_n/(2n-1)$. □

## 5 Quantitative recurrence asymptotics

The qualitative limit (26) does not retain the constant 7/8. For this we use the quantitative Riemann–Hilbert analysis of complex log-Freud weights developed by Lunt, Kriecherbauer, McLaughlin, and von Keyserlingk [6]. We first verify the complex hypotheses.

**Lemma 5.1** (A zero-free complex log-Freud domain)**.** *There exist $\theta \in (0, \pi/2)$ and $\gamma > 0$ such that $Q(z) = \log \mathcal{X}(z)$ has an even analytic branch on*

$$\Omega_{\theta,\gamma} = \{|z| < \gamma\} \cup \{z : |\arg z| < \theta\} \cup \{z : |\arg(-z)| < \theta\}.$$

*On the right cone, uniformly as $z \to \infty$,*

$$Q(z) = \frac{z}{2}\left(\log \frac{z}{2\pi} - 1\right) + \frac{7}{4}\log z + C + O(z^{-1}), \tag{28}$$

$$Q'(z) = \frac{1}{2}\log \frac{z}{2\pi} + \frac{7}{4z} + O(z^{-2}), \qquad Q''(z) = \frac{1}{2z} + O(z^{-2}). \tag{29}$$

*In particular,*

$$\frac{zQ''(z)}{Q'(z)} \longrightarrow 0, \qquad \log z\left(-1 + \frac{zQ'(z)}{Q(z)}\right) \longrightarrow 1. \tag{30}$$

*Thus $Q$ belongs to the class* $\mathrm{CVSLF}(1, 1, \theta, \gamma)$ *of [6].*

*Proof.* Every zero $z_\rho = \rho - 1/2$ of $\mathcal{X}$ satisfies $|\Re z_\rho| < 1/2$. The zero set is discrete and does not meet $\mathbb{R}$, so

$$\eta = \inf_{\mathcal{X}(z)=0} |\Im z| > 0.$$

Choose $\theta$ so small that $\frac{1}{2}\tan\theta < \eta$, and choose $\gamma$ below the distance from 0 to the zero set. The displayed cone-plus-disk domain is star-shaped and zero-free. Hence it admits an analytic logarithm, normalized by $Q(0) = \log \mathcal{X}(0)$. Evenness of $\mathcal{X}$ makes this branch even.

In the right cone, $s = 1/2 + z$ remains in a closed subsector of the right half-plane. Uniform complex Stirling expansions for $\Gamma(s/2)$, together with the uniformly convergent Dirichlet series for $\zeta'/\zeta(s)$, give (28)–(29). Evenness gives the left-cone estimates. The two limits in (30), eventual positivity of $Q'$ on the positive axis, and the local analyticity already proved are precisely the $\mathrm{CVSLF}(1, 1)$ conditions used in [6]. □

The theorem stated in [6] gives $b_n = R_n/2\,(1+O(n^{-1}))$ at hydrodynamic exponent $\rho = 0$. For the present weight, its proof contains a stronger cancellation. Because that refinement is not separately boxed in the source, we record the extraction in detail.

**Proposition 5.2** ($\rho = 0$ recurrence cancellation)**.** *For the reciprocal-$\xi$ weight,*

$$b_n = \frac{R_n}{2}\Big(1 + O(n^{-2})\Big). \tag{31}$$

*Proof.* We use the notation of the Riemann–Hilbert proof in [6, Supplement, Sections on the local parametrices, residual problem, and recurrence extraction]. Their MRS number $\beta_n$ satisfies

$$\frac{1}{2\pi}\int_{-1}^{1} \frac{\beta_n s Q'(\beta_n s)}{\sqrt{1-s^2}}\,\mathrm{d}s = n.$$

Since $Q$ is even, $\beta_n = R_n$.

Specialize the source construction to $p = 1, q = 1, \rho = 0$. There are four observations.

*First*, the shrinking Bessel parametrix at the origin becomes exactly the outside parametrix, $P = N$. The residual matrix therefore has no jump on the origin circle.

*Second*, the lens estimate in the source has the form $\exp[-\Omega((\log n)^c)]$. For $q = 1$, the free contour exponent can be chosen so that $c > 1$. Consequently every lens contribution is smaller than $n^{-M}$ for arbitrary fixed $M$; the exterior jumps are exponentially small.

*Third*, on the two fixed endpoint circles the Airy construction has the uniform expansion

$$\Delta_R(z) = \frac{\Delta_{1,\pm}(z)}{n} + O(n^{-2}). \tag{32}$$

The endpoint density and its analytic derivatives are uniformly bounded, so the remainder is uniform. Thus $\|\Delta_R\|_\infty + \|\Delta_R\|_2 = O(n^{-1})$. For

$$R(z) = I + \frac{R_1}{z} + O(z^{-2}),$$

the small-norm solution gives

$$R_1 = -\frac{1}{2\pi i}\int \Delta_R(y)\,\mathrm{d}y + O(\|\Delta_R\|_2^2). \tag{33}$$

In particular, every entry of $R_1$ is $O(n^{-1})$.

*Fourth*, the recurrence functional uses only $(R_1)_{21} - (R_1)_{12}$ at first order. On the right endpoint, the complete $1/n$ Airy term in that functional is

$$(\Delta_R)_{12} - (\Delta_R)_{21} = -\frac{i}{12n\phi_n(z)}\left(\varphi(z)^\rho - \varphi(z)^{-\rho}\right), \tag{34}$$

with the analogous phase-weighted expression at the left endpoint. Both vanish pointwise when $\rho = 0$. Equations (32)–(33) therefore imply

$$(R_1)_{21} - (R_1)_{12} = O(n^{-2}).$$

The exact source extraction formula reduces at $\rho = 0$ to

$$b_n = \frac{\beta_n}{2}\left(1 + 2i((R_1)_{21} - (R_1)_{12}) + 4(R_1)_{12}(R_1)_{21}\right)^{1/2}.$$

The linear term is $O(n^{-2})$, and the product is quadratic in $R_1 = O(n^{-1})$. Since $\beta_n = R_n$, this proves (31). □

**Remark 5.3.** *The same extraction works for $p = 1, \rho = 0, q > 0$. The restriction $q > 0$ is load-bearing: for $-1 < q \le 0$, the lens estimate proved in [6] need not be polynomially small and does not by itself give an $O(n^{-2})$ recurrence remainder. The supplement pins the source version and audits the origin, endpoint, lens, small-norm, and indexing steps separately.*

**Theorem 5.4** (Shifted Lambert law)**.** *Let $g(x) = \pi x / W_0(2x/e)$. Then*

$$b_n = g(n - 7/8) + O\left(\frac{\log n}{n}\right). \tag{35}$$

*In particular,*

$$\frac{\pi n}{b_n} - W_n = O\left(\frac{\log n}{n}\right), \tag{36}$$

$$\frac{n}{W_n}\left(\frac{\pi n}{b_n} - W_n\right) \longrightarrow \frac{7}{8}, \tag{37}$$

$$n\left(\frac{b_{n+1}}{b_n} - 1\right) \longrightarrow 1, \tag{38}$$

$$(1 + W_n)(b_{n+1} - b_n) \longrightarrow \pi. \tag{39}$$

*Consequently $b_{n+1} > b_n$ for all sufficiently large $n$.*

*Proof.* Since $R_n \asymp n/\log n$, the absolute error in (31) is $O(1/(n \log n))$. Combining proposition 5.2 with (24) proves (35).

Write $a = 7/8$, $m = n - a$, and

$$b_n = g(m) + r_n, \qquad r_n = O(\log n/n).$$

The Lambert identities

$$g'(x) = \frac{\pi}{1 + W_0(2x/e)}, \qquad g(x) = \frac{\pi e}{2} e^{W_0(2x/e)}, \qquad \frac{\mathrm{d}}{\mathrm{d}x} W_0(2x/e) = \frac{W_0(2x/e)}{x(1 + W_0(2x/e))} \tag{40}$$

will be used repeatedly.

The weaker estimate (36) follows immediately from $b_n = g(n)(1 + O(n^{-1}))$. To retain the shift, note that

$$\frac{\pi n}{b_n} - \frac{\pi n}{g(m)} = O\left(\frac{\log^3 n}{n^2}\right).$$

The model term is

$$\frac{\pi n}{g(m)} - W_n = \frac{n}{m} W_m - W_n = \frac{a}{m} W_m + (W_m - W_n).$$

After multiplication by $n/W_n$, the first term tends to $a$, while the second tends to zero by the last identity in (40). This proves (37).

The mean value theorem and (40) give

$$n\left(\frac{g(m+1)}{g(m)} - 1\right) \to 1, \qquad (1 + W_n)(g(m+1) - g(m)) \to \pi.$$

The relative error $r_n/g(m)$ is $O(\log^2 n/n^2)$, and $|r_{n+1} - r_n| = O(\log n/n)$. These are negligible in the two displayed limits, proving (38) and (39). The latter limit is positive, so eventual strict monotonicity follows. ∎

## 6 Polynomial fluctuations and the trace determinant

Let $X_1, \ldots, X_n$ have the probability density obtained by dividing (4) at $t = 0$ by $H_n(0)$. For a real polynomial $P$, set

$$L_n(P) = \sum_{j=1}^{n} P(X_j/b_n)$$

and define the Laurent coefficients

$$\widehat{P}_k = \frac{1}{2\pi} \int_0^{2\pi} P(2\cos\theta) e^{-ik\theta} \,\mathrm{d}\theta. \tag{41}$$

**Theorem 6.1** (Polynomial-statistic CLT). *For every real polynomial $P$,*

$$L_n(P) - \mathbb{E}L_n(P) \Longrightarrow N\left(0, \sum_{k=1}^{\infty} k\widehat{P}_k\widehat{P}_{-k}\right). \tag{42}$$

*Proof.* Push $\mu$ forward under $x \mapsto x/b_n$. The corresponding orthonormal polynomials $q_j^{(n)}(y) = p_j(b_n y)$ satisfy

$$y q_j^{(n)}(y) = \frac{b_{j+1}}{b_n} q_{j+1}^{(n)}(y) + \frac{b_j}{b_n} q_{j-1}^{(n)}(y).$$

By (27), for every fixed integer $r$,

$$\frac{b_{n+r}}{b_n} \longrightarrow 1.$$

Thus the scaled Jacobi matrices have the full-sequence Laurent right limit with symbol $s(z) = z + z^{-1}$. Corollary 2.2 of Breuer–Duits [1] applies to polynomial statistics and gives (42). Their theorem allows the prelimit recurrence matrices to be unbounded; only the Laurent right limit is required to be bounded. □

**Proposition 6.2** (Exact trace normalization). *For $S_n = \sum_{j=1}^{n} X_j$,*

$$\mathbb{E}S_n = 0, \qquad \mathrm{Var}(S_n) = b_n^2, \tag{43}$$

*and*

$$\frac{S_n}{b_n} \Longrightarrow N(0,1). \tag{44}$$

*Proof.* Evenness gives $\mathbb{E}S_n = 0$. For the determinantal projection process,

$$\mathrm{Var}(S_n) = \mathrm{Tr}(P_n J (I - P_n) J P_n).$$

The tridiagonal Jacobi matrix has exactly one edge crossing from ran $P_n$ to its orthogonal complement, namely $e_{n-1} \leftrightarrow e_n$, and its weight is $b_n$. This proves the variance identity. For $P(x) = x$, the only nonzero coefficients in (41) are $\widehat{P}_1 = \widehat{P}_{-1} = 1$, so theorem 6.1 gives (44). □

**Corollary 6.3** (Gaussian Hankel window). *For every $U > 0$,*

$$\sup_{|u| \le U} \left| \frac{H_n(u/b_n)}{H_n(0)} - e^{-u^2/2} \right| \longrightarrow 0. \tag{45}$$

*Consequently $H_n(t) > 0$ for $|t| \le U/b_n$ once $n$ is sufficiently large, and*

$$\frac{1}{b_n} \sim \frac{W_0(2n/e)}{\pi n} \sim \frac{\log n}{\pi n}.$$

*Proof.* By (5), the determinant ratio at $u/b_n$ is the characteristic function of $S_n/b_n$, so (44) gives pointwise convergence. Moreover,

$$\mathbb{E}|S_n/b_n| \le \left(\mathbb{E}(S_n/b_n)^2\right)^{1/2} = 1$$

by (43); hence these characteristic functions are uniformly Lipschitz. A finite-net argument upgrades pointwise convergence to (45). The characteristic functions are real because $S_n$ has an even law, and the Gaussian limit is bounded below by $e^{-U^2/2} > 0$ on the compact interval. $\square$

# 7 The Ullman particle and zero law

Define

$$c_n = \frac{\pi n}{W_0(2n/e)}.$$

Then $b_n/c_n \to 1$, and $c_n$ is regularly varying with index one:

$$\frac{c_{\lfloor tn \rfloor}}{c_n} \longrightarrow t \qquad (t > 0).$$

**Theorem 7.1** (Global particle and zero limit)**.** *Let*

$$\nu_n^{\mathrm{OPE}} = \frac{1}{n}\sum_{j=1}^{n} \delta_{X_j/b_n}, \qquad \nu_n^{\mathrm{zero}} = \frac{1}{n}\sum_{p_n(x_{j,n})=0} \delta_{x_{j,n}/b_n}.$$

*Then*

$$\nu_n^{\mathrm{OPE}} \Longrightarrow \nu_{\mathrm{U}} \quad \textit{weakly in probability}, \qquad \nu_n^{\mathrm{zero}} \Longrightarrow \nu_{\mathrm{U}} \quad \textit{weakly},$$

*where*

$$\mathrm{d}\nu_{\mathrm{U}}(x) = \frac{1}{2\pi}\operatorname{arcosh}\left(\frac{2}{|x|}\right)\mathbf{1}_{\{0<|x|<2\}}\,\mathrm{d}x. \tag{46}$$

*Its moments are*

$$\int x^{2m+1}\,\mathrm{d}\nu_{\mathrm{U}}(x) = 0, \qquad \int x^{2m}\,\mathrm{d}\nu_{\mathrm{U}}(x) = \frac{\binom{2m}{m}}{2m+1}. \tag{47}$$

*Proof.* In the notation of Liu–Wang–Yan [4], the scaled off-diagonal recurrence limit is 1, the diagonal limit is 0, and the contraction has regular-variation index $\lambda = 1$. Their moment formula gives

$$M_k = \frac{1}{1+k}\,\mathrm{CT}(z + z^{-1})^k.$$

This is zero for odd $k$, while $\mathrm{CT}(z + z^{-1})^{2m} = \binom{2m}{m}$, proving (47). Their expected-density theorem and the Nevai–Dehesa zero theorem [7], as recorded in [4, Remark 1.8], give respectively the expected particle limit and deterministic zero limit. Replacing $c_n$ by $b_n$ does not alter either limit.

To identify the measure, let $T$ be uniform on $[0, 1]$, let $\Theta$ be uniform on $[0, 2\pi]$, and take them independent. The variable $2T\cos\Theta$ has the moments in (47). Mixing the conditional arcsine density gives, for $0 < |x| < 2$,

$$\int_{|x|/2}^{1} \frac{\mathrm{d}t}{\pi\sqrt{4t^2 - x^2}} = \frac{1}{2\pi}\operatorname{arcosh}\frac{2}{|x|},$$

which proves (46).

It remains to upgrade expected particle convergence. By theorem 6.1, for each polynomial $P$,

$$\sum_{j=1}^{n} P(X_j/b_n) - \mathbb{E}\sum_{j=1}^{n} P(X_j/b_n) = O_{\mathbb{P}}(1).$$

After division by $n$, every empirical polynomial moment therefore converges in probability to (47). Expected even moments of arbitrarily high order remain bounded, which gives tightness in probability. The compactly supported moment limit is determinate, so the standard subsequence argument yields weak convergence in probability. □

# 8 Hankel free energy

**Theorem 8.1** (Zero-time partition asymptotic)**.** *The exact product* (7) *satisfies*

$$\log H_n(0) = n^2 \left( \log n - \log\log n + \log \pi - \frac{3}{2} \right) + o(n^2). \tag{48}$$

*Equivalently,*

$$H_n(0)^{1/n^2} \sim \frac{\pi n}{e^{3/2}\log n}. \tag{49}$$

*Proof.* By (7) and (27),

$$\log H_n(0) = n\log Z + 2\sum_{j=1}^{n-1}(n-j)\left(\log \pi + \log j - \log W_0(2j/e) + o(1)\right).$$

The weighted Cesaro implication

$$\varepsilon_j \to 0 \quad\Longrightarrow\quad \sum_{j<n}(n-j)\varepsilon_j = o(n^2)$$

justifies the last term. The constant contribution is

$$2\log\pi\sum_{j<n}(n-j) = n^2\log\pi + o(n^2).$$

A Riemann-sum calculation gives

$$2\sum_{j<n}(n-j)\log j = n^2\log n + 2n^2\int_0^1 (1-t)\log t\,\mathrm{d}t + o(n^2) = n^2\log n - \frac{3}{2}n^2 + o(n^2).$$

Finally, put $J_n = \lfloor n/(\log n)^2 \rfloor$. Since $\log W_0(2j/e) = O(\log n)$ for $j \le J_n$, this initial range contributes $O(nJ_n\log n) = o(n^2)$. Uniformly for $j > J_n$,

$$\log W_0(2j/e) = \log\log j + o(1) = \log\log n + o(1).$$

Therefore

$$2\sum_{j<n}(n-j)\log W_0(2j/e) = n^2\log\log n + o(n^2).$$

The term $n\log Z$ is $o(n^2)$. This proves (48), and exponentiation gives (49). □

# 9 Logical scope and off-axis counterfeits

The preceding theorems depend on the real-axis potential and, for the quantitative recurrence estimate, on an analytic logarithm in a narrow cone about the real axis. These data do not determine the entire zero divisor. We make this explicit with three examples.

**Proposition 9.1** (Counterfeits)**.** *The following even entire functions are positive on $\mathbb{R}$ and have zeros off the imaginary axis.*

(a) $M_0(z) = 2 + \cosh z$ *satisfies*

$$(\log M_0)''(x) > 0, \qquad (\log M_0)'''(x) = -\frac{2(\cosh x - 1)\sinh x}{(2+\cosh x)^3} < 0 \quad (x > 0).$$

*Thus the GHS–Jensen mechanism and the order-one Freud/CLT mechanism coexist with the zeros*

$$z = \pm \operatorname{arcosh} 2 + (2k+1)\pi i.$$

(b) $M_1(z) = \mathcal{X}(z)(z^4+1)$ *has the same leading recurrence scale, Ullman law, and $n^2$ free energy as $\mathcal{X}$, while $(1+i)/\sqrt{2}$ is a zero.*

(c) *Put $P(z) = z^4 + 6z^2 + 25$. For every sufficiently large $A > 0$,*

$$M_A(z) = \mathcal{X}(z) + AP(z)$$

*is positive on $\mathbb{R}$, has zeros near $\pm(1+2i), \pm(1-2i)$, and has the same complex log-Freud expansion (28) to every algebraic order in a sufficiently narrow real-axis cone. In particular, it retains the $7/8$ MRS shift and the recurrence consequences of theorem 5.4.*

*Proof.* For (a), direct differentiation gives the displayed inequalities, and $\cosh(\pm \operatorname{arcosh} 2 + (2k+1)\pi i) = -2$.

For (b), the factor $x^4 + 1$ is positive on $\mathbb{R}$, while

$$\frac{\mathrm{d}}{\mathrm{d}x} \log(x^4+1) = \frac{4}{x} + O(x^{-5}).$$

This perturbation changes only lower-order MRS data and preserves the leading $n/\log n$ recurrence scale, its regular variation, and the free-energy coefficient through order $n^2$.

For (c),

$$P(x) = (x^2+3)^2 + 16 > 0 \qquad (x \in \mathbb{R}),$$

and the four listed zeros of $P$ are simple. Rouché's theorem on small circles around them gives nearby zeros of $M_A$ when $A$ is large. In a narrow cone about either real axis, $\mathcal{X}(z)$ dominates every polynomial as $|z| \to \infty$. Hence $AP(z)/\mathcal{X}(z)$ is smaller than every algebraic power there. After narrowing the cone to avoid the finitely many compact zeros, $\log M_A$ has the same asymptotic expansion as $\log \mathcal{X}$, and all claims follow. □

**Remark 9.2.** *The proposition is not a statement about approximation to $\xi$. Its purpose is logical: any argument using only the premises retained by one of these counterfeits cannot imply that all zeros of $\mathcal{X}$ are imaginary. In particular, the shrinking determinant positivity window in corollary 6.3 does not yield positivity at a fixed nonzero time or an all-order Pólya-frequency theorem.*

## 10 Reproducibility and certified constants

The analytic results above use computation only in (19), which closes the finite-index global floor. The accompanying artifact also audits all normalization constants and algebraic reductions used in the asymptotic statements.

The interval certificate uses Arb ball arithmetic at 90 decimal digits. It encloses

$$\begin{aligned} Q''(0) &= 0.046209986230837941577867620860678028 0\ldots, \\ Z &= 24.007072252007373670587893361 43\ldots, \\ \int_{\mathbb{R}} \frac{x^2}{\mathcal{X}(x)}\,\mathrm{d}x &= 567.946364978973522191318737 5\ldots, \\ b_1^2 &= 23.657460560668082246008241492\ldots. \end{aligned}$$

The integration is regularized across the removable zeta pole at $s = 1$, performed directly through $x = 80$, and completed by a geometric tail bound obtained from the step-two completed-zeta recurrence.

Eight additional deterministic checks verify:

- the concavity identity (12) and the elementary endpoint margins in lemma 3.3;
- the coefficients $7/4$, $1/48$, and $7/8$, the exact integrals (25), and Lambert-$W$ inversion;
- the $\rho = 0$ recurrence-cancellation algebra and all three normalized limits in theorem 5.4;
- the crossing-edge variance, Laurent Fourier coefficients, and compact-window normalization;
- the Ullman moments and density normalization;
- the Gram/Jacobi product and the free-energy constant $-3/2$; and
- the pinned source anchors and complete order ledger behind the $\rho = 0$ recurrence cancellation, including the sharp $q > 0$ boundary in remark 5.3.

The supplement contains the exact Python environment lock, verifier sources, immutable JSON outputs, a SHA-256 manifest, and a runner. From the supplement root, the complete audit is

```
uv sync --frozen
uv run --frozen python verify_all.py
```

No network access is required after the locked environment has been materialized.

## Data availability

All code and derived verification records needed to reproduce the computer-assisted statements are included in the accompanying reproducibility supplement. The paper otherwise uses no empirical data.

## Declaration

The author is identified publicly as Kenan. Affiliation, journal selection, and submission metadata are omitted from this version.

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