

Explicit zero-free regions for the Riemann zeta-function

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Explicit zero-free regions for the Riemann zeta-function

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A thesis in fulfilment of the requirements for the degree of
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Abstract

Over the course of the last century, the Riemann zeta-function has played a pivotal role in furthering our understanding of the distribution of prime numbers. Of particular importance are the locations of its complex-valued zeroes — a topic which is currently poorly understood and a hurdle to establishing many results in analytic number theory. It is known that all such zeroes lie inside the critical strip, the set of complex numbers s for which $0 < \operatorname{Re} s < 1$. The famous Riemann hypothesis asserts that these zeroes in fact lie on the vertical line $\operatorname{Re} s = 1/2$; however, at present the best available techniques can only exclude zeroes from certain subsets of the critical strip, known as zero-free regions. In this thesis, we develop a collection of analytic and computational tools to refine existing approaches to construct zero-free regions. These tools allow us to prove a series of explicit zero-free regions larger than those previously known.

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Publications

The following papers were either published or accepted for publication in journals during my PhD candidature. Ideas from some of these publications appear in Chapter 3 and 4 of this thesis and are cited where appropriate.

- D. Johnston and A. Yang. “Some explicit estimates for the error term in the prime number theorem”. In: *J. Math. Anal. Appl.* 527.2, 127460 (2023).
- G. A. Hiary, D. Patel, and A. Yang. “An improved explicit estimate for $\zeta(1/2 + it)$ ”. In: *J. Number Theory* 256 (2024), pp. 195–217.
- M. J. Mossinghoff, T. S. Trudgian, and A. Yang. “Explicit zero-free regions for the Riemann zeta-function”. In: *Res. Number Theory* 10, 11 (2024).
- A. Yang. “Explicit bounds on $\zeta(s)$ in the critical strip and a zero-free region”. In: *J. Math. Anal. Appl.* 534.2, 128124 (2024).
- D. Patel and A. Yang. “An explicit sub-Weyl bound for $\zeta(1/2 + it)$ ”. In: *J. Number Theory* 262 (2024), pp. 301–334.
- C. Bellotti and A. Yang. “On the generalised Dirichlet divisor problem”. In: *Bull. Lond. Math. Soc.* 56.5 (2024), pp. 1859–1878.
- T. S. Trudgian and A. Yang. “Toward optimal exponent pairs”. In: *Math. Comput.* 94 (2025), pp. 1467–1502.
- G. A. Hiary, N. Leong, and A. Yang. “Explicit bounds for the Riemann zeta-function on the 1-line”. In: *Funct. Approx. Comment. Math.* 73.1 (2025), pp. 53–89.
- T. Tao, T. S. Trudgian, and A. Yang. “New exponent pairs, zero density estimates, and zero additive energy estimates: a systematic approach”. In: *Math. Comput.* (to appear) (2025).

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Chapter 1

Introduction

Let $s = \sigma + it$ denote a complex number, and let $\zeta(s)$ denote the *Riemann zeta-function*, defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for $\sigma > 1$, and by its analytic continuation in the rest of the complex plane, except at $s = 1$ where there is a simple pole. We refer to the set of complex numbers s for which $0 < \operatorname{Re} s < 1$ as the *critical strip*. The behaviour of $\zeta(s)$ in the critical strip is both mysterious and intimately connected to the distribution of prime numbers. In particular, an important open problem in analytic number theory is the location and distribution of complex numbers ρ for which $\zeta(\rho) = 0$, collectively known as the *zeroes* of the Riemann zeta-function.

The zeroes of $\zeta(s)$ fall into two categories. There are infinitely many real *trivial zeroes* occurring at $s = -2n$ for any positive integer n ; these zeroes are well-understood. This thesis instead focuses on the locations of the less-understood *non-trivial zeroes*, which are complex valued. One motivation for studying such zeroes is their astonishing connection to the prime numbers, as demonstrated through the “explicit” formula [Man95]

$$\sum_{n \leq x} \Lambda(n) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \log 2\pi - \frac{1}{2} \log(1 - x^{-2}),$$

for $x > 0$ not coinciding with a prime power, where the sum on the right is over all non-trivial zeroes of $\zeta(s)$, and $\Lambda(n)$ is the von Mangoldt function, defined as $\log p$ if $n = p^m$ for some prime number p and positive integer m , and 0 otherwise.

However, despite over a century of effort initiated by Riemann's famous memoir [Rie59], relatively little is known about the precise locations of the non-trivial zeroes. By the classical Euler product formula

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}} \quad (\operatorname{Re} s > 1)$$

where p runs through the prime numbers, one can see that there are no zeroes ρ with $\operatorname{Re} \rho > 1$. In addition, by the functional equation (see e.g. Titchmarsh [Tit86, Ch. II])

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s),$$

it is evident that there are also no zeroes with $\operatorname{Re} \rho < 0$ other than those arising from the poles of the gamma function, which are precisely the trivial zeroes. Therefore, all non-trivial zeroes reside inside the strip $0 \leq \operatorname{Re} s \leq 1$. We can also observe from the functional equation that if ρ is a non-trivial zero, then so is $1 - \bar{\rho}$, i.e. non-trivial zeroes exist in pairs that are symmetric about the vertical line $\operatorname{Re} s = 1/2$, commonly known as the *critical line*. Furthermore, since $\overline{\zeta(s)} = \zeta(\bar{s})$, the set of non-trivial zeroes is also symmetric about the real axis. It is also known that there are infinitely many such zeroes [Har14]; in fact precise growth rates are known: the number of non-trivial zeroes ρ with $\operatorname{Im} \rho \in [-T, T]$ is $(1/\pi + o(1))T \log T$ as $T \rightarrow \infty$ [Man05].

Riemann's paper contained a conjecture that all non-trivial zeroes in fact lie on the critical line; this later became known as the Riemann hypothesis, a famous open problem in pure mathematics. The truth of the Riemann hypothesis will have far-reaching consequences in many fields, including for instance the prime number estimate [Sch76]

$$|\pi(x) - \operatorname{li}(x)| < \frac{1}{8\pi} x^{1/2} \log x, \quad \operatorname{li}(x) := \int_2^x \frac{dt}{\log t},$$

for $x \geq 2657$, where $\pi(x)$ denotes the number of primes no greater than x . While a proof of the Riemann hypothesis appears far out of reach, a number of partial results

have been established. It is known that a positive proportion of zeroes lie on the critical line [Sel42], and that the proportion is at least $5/12$ [Pra+20]. Computationally, the first 10^{13} zeroes have been verified to lie on the critical line [PT21]. A number of zero-density estimates have been proved, which bound the quantity $N(\sigma, T)$ counting the number of zeroes $\rho = \beta + i\gamma$ with $\beta \geq \sigma > 1/2$ and $|\gamma| \leq T$. Lastly, it is known that there are no zeroes inside certain regions within the critical strip; we refer to these regions as *zero-free regions*.

The goal of this thesis is to enlarge the known explicit zero-free regions by refining a number of methods that we will review over the next few chapters. Here we refer to a result as “explicit” if all implied constants are specified (including, say, those constants implied by requiring a parameter to be “sufficiently large”). In the next few sections, we review the historical progression of results in this area and then present the new results that will be proved in this thesis.

1.1 Classical zero-free regions

In 1896, Hadamard [Had96] and de la Vallée Poussin [VP96] showed independently that $\zeta(s)$ has no zeroes on the vertical line $\operatorname{Re} s = 1$ (and thus also $\zeta(s) \neq 0$ on $\operatorname{Re} s = 0$). This represented the first “non-trivial” result regarding the location of zeta zeroes, and led to the proof of the celebrated prime number theorem that

$$\pi(x) \sim \operatorname{li}(x)$$

as $x \rightarrow \infty$. In 1899, de la Vallée Poussin [VP99] adapted the method to show that if t is sufficiently large, then in fact $\zeta(\sigma + it) \neq 0$ for

$$\sigma > 1 - \frac{A}{\log t}$$

for some absolute constant $A > 0$. We shall refer to this as the *classical zero-free region*, and this implies the estimate

$$\pi(x) - \operatorname{li}(x) \ll x \exp\left(-c(\log x)^{1/2}\right)$$

for some constant $c > 0$. Here, we provide a sketch of the proof of de la Vallée Poussin's result, which will be useful to facilitate a comparison with the various refinements to the method that followed. One begins by writing

$$-\operatorname{Re} \frac{\zeta'(s)}{\zeta(s)}$$

in two ways: first as a Dirichlet series (for $\operatorname{Re} s > 1$), and second as an expansion obtained via the Hadamard product formula (see e.g. Davenport [Dav00, Ch. 12]). This gives

$$\operatorname{Re} \sum_{n \geq 1} \frac{\Lambda(n)}{n^s} = -\frac{1}{2} \log \pi + \operatorname{Re} \frac{1}{2} \frac{\Gamma'}{\Gamma} \left(\frac{s}{2} + 1 \right) + \operatorname{Re} \frac{1}{s-1} - \operatorname{Re} \sum_{\rho} \frac{1}{s-\rho}, \quad (1.1)$$

for $\operatorname{Re} s > 1$, where the sum is over non-trivial zeroes of zeta (here and thereafter, zeroes are summed in conjugate pairs and counted with multiplicity). Suppose now that $\rho_0 = \beta + it$ is a non-trivial zero off the critical line. For some $\sigma > 1$ to be chosen later, we evaluate (1.1) at $s = \sigma, \sigma + it$ and $\sigma + 2it$. Routine calculations give (as $t \rightarrow \infty$)

$$\sum_{n \geq 1} \frac{\Lambda(n)}{n^\sigma} = \frac{1}{\sigma-1} + O(1),$$

$$\operatorname{Re} \sum_{n \geq 1} \frac{\Lambda(n)}{n^{\sigma+it}} = -\operatorname{Re} \sum_{\rho} \frac{1}{\sigma+it-\rho} + O(\log t), \quad (1.2)$$

$$\operatorname{Re} \sum_{n \geq 1} \frac{\Lambda(n)}{n^{\sigma+2it}} = -\operatorname{Re} \sum_{\rho} \frac{1}{\sigma+2it-\rho} + O(\log t), \quad (1.3)$$

where all implied constants are absolute. Meanwhile, in light of the non-negative trigonometric polynomial

$$3 + 4 \cos \theta + \cos 2\theta = 2(1 + \cos \theta)^2 \geq 0, \quad (1.4)$$

one has

$$\begin{aligned} 0 &\leq \sum_{n \geq 1} \frac{\Lambda(n)}{n^\sigma} (3 + 4 \cos(t \log n) + \cos(2t \log n)) \\ &= \frac{3}{\sigma-1} - 4 \operatorname{Re} \sum_{\rho} \frac{1}{\sigma+it-\rho} - \operatorname{Re} \sum_{\rho} \frac{1}{\sigma+2it-\rho} + O(\log t). \end{aligned}$$

Next we make use of the fact that the terms appearing in the sums are one-signed, since $\sigma > 1 > \operatorname{Re} \rho$ and

$$\operatorname{Re} \frac{1}{z} > 0 \quad (\operatorname{Re} z > 0). \quad (1.5)$$

We use this to drop the entire second sum, and also all terms except for $\rho = \rho_0$ in the first sum. This gives

$$0 \leq \frac{3}{\sigma - 1} - \frac{4}{\sigma - \beta} + O(\log t), \quad (1.6)$$

where the implied constant can be made explicit. At this point we choose

$$\sigma = 1 + \frac{c}{\log t}$$

where c is a suitable positive constant, which, upon rearranging, gives

$$\beta \leq 1 - \frac{A}{\log t}$$

for some absolute constant $A > 0$ if t is sufficiently large.

In fact in [VP99, Theorem 31] it was shown that $A = (30.46)^{-1} = 0.03282 \dots$ is permissible for t sufficiently large. Subsequent refinements over the course of the last century steadily increased the size of A but most still followed essentially the same argument. The historical progression in this constant is recorded in Table 1.1, which we review next in chronological order.

Table 1.1: Historical progress in the constant A .

Year	Reference	A
1899	de la Vallée Poussin [VP99]	$(30.46)^{-1} = 0.0328 \dots$
1909	Landau [Lan09]	$(18.524)^{-1} = 0.0539 \dots$
1938	Westphal [Wes38]	$(17.537)^{-1} = 0.0570 \dots$
1941	Rosser [Ros41]	$(17.72)^{-1} = 0.0564 \dots$
1962	Rosser–Schoenfeld [RS62]	$(17.517)^{-1} = 0.0570 \dots$
1970	Stechkin [Ste70]	$(9.65)^{-1} = 0.103 \dots$
1975	Rosser–Schoenfeld [RS75]	$(9.646)^{-1} = 0.103 \dots$
1977	Kondrat’ev [Kon77]	$(9.548)^{-1} = 0.104 \dots$
2002	Ford [For02b]	$(8.463)^{-1} = 0.118 \dots$
2005	Kadiri [Kad05]	$(5.697)^{-1} = 0.175 \dots$
2014	Jang–Kwon [JK14]	$(5.684)^{-1} = 0.175 \dots$
2014	Mossinghoff–Trudgian [MT14]	$(5.573)^{-1} = 0.179 \dots$
2022	Mossinghoff–Trudgian–Yang [MTY24]	$(5.559)^{-1} = 0.179 \dots$
2025	Theorem 1.1	$(4.862)^{-1} = 0.205 \dots$

The first refinement, due to Landau (1909) [Lan09, page 321], was based on the observation that one may replace the trigonometric polynomial (1.4) with any polynomial of the form

$$P(\theta) = \sum_{0 \leq k \leq K} a_k \cos k\theta$$

provided that $P(\theta) \geq 0$, $a_k \geq 0$ and $a_1 > a_0$. Landau chose the polynomial $5 + 8 \cos \theta + 4 \cos 2\theta + \cos 3\theta$ to obtain $A = (18.524)^{-1}$. This was subsequently improved to $A = (17.537)^{-1}$ by Westphal (1938) [Wes38] and $A = (17.517)^{-1}$ by Rosser–Schoenfeld (1962) [RS62], using different polynomials.

The next idea, due to Stechkin (1970) [Ste70], was to make use of the fact that non-trivial zeroes of $\zeta(s)$ exist in pairs symmetric about the critical line. That is, if ρ is a non-trivial zero, then so is $1 - \bar{\rho}$. Central to Stechkin’s approach was the observation that if $F(z) = z^{-1} - \kappa(z + \delta)^{-1}$, where $\kappa = 1/\sqrt{5}$ and $\delta = (1 - \sigma + \sqrt{1 + 4\sigma^2})/2$, then for any pair of non-trivial zeroes ρ and $\rho' = 1 - \bar{\rho}$ one has

$$\operatorname{Re} F(s - \rho) + \operatorname{Re} F(s - \rho') \geq 0. \quad (1.7)$$

This achieves the same outcome as the inequality (1.5); however, one may now repeat the classical argument with (1.1) replaced by

$$\begin{aligned} \operatorname{Re} \sum_{n \geq 1} \frac{\Lambda(n)}{n^s} \left(1 - \frac{\kappa}{n^\delta}\right) &= -\frac{1 - \kappa}{2} \log \pi + \operatorname{Re} \frac{1}{2} \left(\frac{\Gamma'}{\Gamma} \left(\frac{s}{2} + 1 \right) - \kappa \frac{\Gamma'}{\Gamma} \left(\frac{s + \delta}{2} + 1 \right) \right) \\ &\quad + \operatorname{Re} F(s - 1) - \operatorname{Re} \sum_{\rho} F(s - \rho). \end{aligned} \quad (1.8)$$

The partial cancellation of the Γ'/Γ terms ultimately increases A by a factor of $(1 - \kappa)^{-1}$ for sufficiently large t . Using this method Stechkin obtained $A = (9.65)^{-1}$. This was later improved to $A = (9.646)^{-1}$ by Rosser–Schoenfeld (1975) [RS75] and to $A = (9.548)^{-1}$ by Kondrat’ev (1977) [Kon77], using different trigonometric polynomials.

From the modern perspective, both (1.1) and (1.8) can be viewed as examples of *zero-detectors*. Here, a zero-detector is simply a complex function $Q(s)$ that is large when s is close to a non-trivial zero, and which can be represented both as a Dirichlet series as well as an expansion involving the zeroes of $\zeta(s)$. In 1992, Heath-Brown [HB92a] considered the problem of choosing the most favourable zero-detector of the form

$$\begin{aligned} \operatorname{Re} \sum_{n \geq 1} \frac{\Lambda(n)}{n^s} f(\log n) &= \frac{f(0)}{2} \operatorname{Re} \frac{\Gamma'}{\Gamma} \left(\frac{s}{2} + 1 \right) + \operatorname{Re} F(s - 1) - \sum_{\rho} \operatorname{Re} F(s - \rho) \\ &\quad + \text{error terms,} \end{aligned} \quad (1.9)$$

where $f(x)$ is any non-negative, compactly supported function satisfying certain regularity conditions, and F is now the Laplace transform of f . Ignoring temporarily the requirement that f be compact, this zero-detector generalizes both (1.1) and (1.8) (which take $f(x) = 1$ and $f(x) = 1 - \kappa e^{-\delta x}$ respectively). Since such zero-detectors possess a Hadamard-type expansion (commonly known as Weil’s explicit formula [Wei52]), one may use them directly in place of (1.1). A calculus-of-variations argument is then used to derive specific choices of f . Importantly, such zero-detectors have a larger radius of convergence than (1.1), which allows one to take s within the critical strip (instead of $\sigma > 1$). Although [HB92a] did not derive an explicit zero-free region for zeta (the focus was on other L -functions), this marked a theoretical breakthrough and the same choices of f appear in all subsequent works.

In 2002, Ford [For02b] developed a new zero-detector with the same Dirichlet series representation as Heath-Brown’s zero-detector; however it replaces the usual Hadamard expansion with a Jensen-type formula:

$$\begin{aligned} \operatorname{Re} \sum_{n \geq 1} \frac{\Lambda(n)}{n^s} f(\log n) &= \frac{f(0)}{4\lambda} \int_{-\infty}^{\infty} \frac{\log |\zeta(s - \lambda + 2\lambda i u / \pi)| - \log |\zeta(s + \lambda + 2\lambda i u / \pi)|}{(\cosh u)^2} du \\ &\quad + \operatorname{Re} F(s - 1) - \sum_{\substack{\rho \\ \operatorname{Re} \rho \geq \sigma - \lambda}} \operatorname{Re} F(s - \rho) + \text{error terms}, \end{aligned} \tag{1.10}$$

for suitable $\lambda > 0$. This formula is characteristic of Landau’s “local” approach (see §1.2 below), which in principle only uses properties of zeta close to the point s (as opposed to de la Vallée Poussin’s “global” approach that involve e.g. sums over all zeta zeroes). As with most Jensen-type formulas, Ford’s zero detector allows one to pass from bounds on the size of a function to knowledge about its zeroes, and requires no information about the specific analytic properties of $\zeta(s)$. When combined with an explicit subconvexity estimate of $\zeta(s)$ on the critical line, Ford obtained $A = (8.463)^{-1}$.

Neither of the two previous treatments made substantial use of the fact that non-trivial zeroes exist in pairs about the critical line — this is a limitation of the “local” approach since one cannot apply (1.7) as the sum in (1.10) contains at most one zero from each

pair. Nevertheless, Kadiri (2005) [Kad05] was able to combine the techniques of Stechkin and Heath-Brown by using a zero-detector of the form

$$\sum_{n \geq 1} \frac{\Lambda(n)}{n^s} \left(1 - \frac{\kappa'}{n^{\delta'}}\right) f(\log n) = R(s) - \kappa' R(s + \delta'), \quad (1.11)$$

where $R(s)$ is the expression on the right side of (1.9), κ' and δ' are positive constants and f is a smoothing function from [HB92a]. Along with some other refinements, Kadiri obtained $A = (5.697)^{-1}$. Later, Jang-Kwon (2014) [JK14] revisited Kadiri's approach (1.11) with various other choices of f found in [HB92a], as well as a function studied by Xylouris [Xyl11], to obtain $A = (5.684)^{-1}$.

The next source of improvement was computational in nature. In 2014, Mossinghoff-Trudgian [MT14] used simulated annealing, a probabilistic optimization algorithm, to find favourable non-negative trigonometric polynomials of high degree (previous treatments used polynomials of degree at most four, with the exception of [Kon77] who used a degree eight polynomial). This search algorithm was particularly well-suited for (approximate) global optimization problems when the objective function has many suboptimal local maxima. A large scale computation found a 16th degree polynomial, which, together with other numerical refinements, produced $A = (5.573)^{-1}$. Figure 1.1 compares this polynomial to some earlier polynomials. Later, Mossinghoff-Trudgian-Yang (2024) [MTY24] slightly modified the argument to obtain $A = (5.559)^{-1}$.

In this thesis, we will combine the computational and analytic tools of previous works to show that $A = (4.862)^{-1}$ is admissible (Theorem 1.1 below). This is the goal of Chapter 2.

1.2 Asymptotically larger zero-free regions

A different approach to deriving zero-free regions was developed in 1924 by Landau [Lan24] (see Titchmarsh [Tit86, pages 56–59] for an exposition). Roughly speaking, one first relates the zeroes of $\zeta(s)$ on some domain $\mathcal{D} \subset \mathbb{C}$ to the modulus of $\zeta(s)$ on $\mathcal{D}$. Then, using known upper-bounds on $|\zeta(s)|$, one arrives at a contradiction if any zero is too close to the line

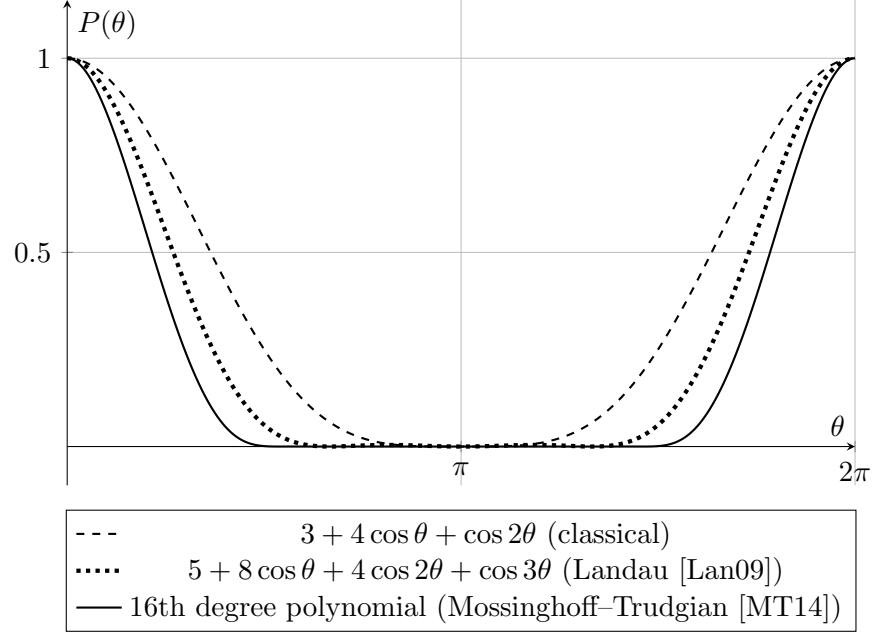


Figure 1.1: Comparison of trigonometric polynomials $P(\theta)$ (normalized so that $P(0) = 1$).

$\sigma = 1$. The first step can be accomplished using Jensen’s formula or another similar device (such as Ford’s zero-detector (1.10)). The second step typically requires an estimate of $\zeta(s)$ inside the critical strip, which can be obtained via the (often difficult) analysis of certain exponential sums.

One limitation of de la Vallée Poussin’s approach is that $\operatorname{Re} \Gamma' / \Gamma(s/2 + 1) \gg \log t$, so one can only obtain zero-free regions of classical type, i.e. $\sigma > 1 - A / \log t$ for some constant A . On the other hand, by appealing to certain non-trivial estimates of the exponential sum

$$S(N, t) := \sum_{N < n \leq 2N} n^{-it},$$

one can obtain bounds on $\zeta(s)$ that are sharp enough to generate asymptotically larger zero-free regions using Landau’s method. Littlewood [Lit22] in 1922 used the Weyl–van der Corput method to derive an estimate of the form

$$S(N, t) \ll N^{1-k/(2^k-2)} t^{1/(2^k-2)} + N^{1-2^{2-k}+k/(2^k-2)} t^{-1/(2^k-2)}$$

uniformly for integers $k \geq 3$, which was used to show that

$$\zeta(1 - k/(2^k - 2) + it) \ll t^{1/(2^k-2)} \log t \quad (t \rightarrow \infty)$$

and consequently that $\zeta(\sigma + it) \neq 0$ for t sufficiently large and

$$\sigma > 1 - A \frac{\log \log t}{\log t} \quad (1.12)$$

where $A > 0$ is an absolute constant. This also implies the prime number estimate

$$\pi(x) - \text{li}(x) \ll x \exp(-c\sqrt{\log x \log \log x}) \quad (x \rightarrow \infty)$$

for some constant $c > 0$.

In 1935, Vinogradov [Vin35] developed a method for estimating averages of certain exponential sums, which became known later as Vinogradov's mean value theorem. The method was successively refined by multiple authors, including Linnik [Lin43], Karatsuba [Kar73], Stechkin [Ste75], Wooley [Woo12; Woo16] and Bourgain–Demeter–Guth [BDG16]. In its modern form, Vinogradov's method implies (amongst other consequences) an estimate of the form

$$S(N, t) \ll N^{1-c(\log N / \log t)^2}$$

for some absolute constant $c > 0$, and also the following bound (due to Richert [Ric67])

$$\zeta(\sigma + it) \ll t^{B(1-\sigma)^{3/2}} (\log t)^{2/3} \quad (1/2 \leq \sigma \leq 1) \quad (1.13)$$

where $B > 0$ is an absolute constant. In 1938, Chudakov [Chu38] used an early form of Vinogradov's mean value theorem to show that $\zeta(\sigma + it) \neq 0$ for

$$\sigma > 1 - \frac{1}{(\log t)^{3/4+o(1)}}$$

as $t \rightarrow \infty$. Then, in 1958, Vinogradov [Vin58] and Korobov [Kor58] independently proved¹ that $\zeta(\sigma + it) \neq 0$ for t sufficiently large and

$$\sigma > 1 - \frac{A}{(\log t)^{2/3} (\log \log t)^{1/3}} \quad (1.14)$$

which is currently the form of the largest known zero-free region as $t \rightarrow \infty$. This region also implies the prime number estimate

$$\pi(x) - \text{li}(x) \ll x \exp(-c(\log x)^{3/5} (\log \log x)^{-1/5}) \quad (x \rightarrow \infty)$$

¹Both proofs in fact claim the larger zero-free region $\sigma > 1 - A(\log t)^{-2/3}$; however their methods appear to only produce a region of strength (1.14).

for some constant $c > 0$.

A number of explicit versions of (1.12) and (1.14) have been derived, which we will review in Chapters 3 and 4. In the explicit setting, the values of A attainable for such zero-free regions are often substantially smaller than those found in the classical zero-free region. For many applications in explicit number theory, the classical zero-free region remains the largest for ranges of t of interest. Nevertheless, in Chapter 3 and 4 of this thesis, we derive improved explicit values of A for (1.12) and (1.14) respectively (Theorem 1.2 and Theorem 1.3 below).

1.3 New results

The aim of this thesis is to establish the following zero-free regions, which, to the best of the author's knowledge, are new:

Theorem 1.1. *One has $\zeta(\sigma + it) \neq 0$ if $t \geq 2$ and*

$$\sigma > 1 - \frac{1}{4.862 \log t}.$$

Theorem 1.2. *One has $\zeta(\sigma + it) \neq 0$ if $t \geq 3$ and*

$$\sigma > 1 - \frac{\log \log t}{19.62 \log t}.$$

Theorem 1.3. *One has $\zeta(\sigma + it) \neq 0$ if $t \geq 3$ and*

$$\sigma > 1 - \frac{1}{51.34 (\log t)^{2/3} (\log \log t)^{1/3}}.$$

The proofs of these theorems are the subject of Chapters 2, 3 and 4 respectively. In Platt–Trudgian (2021) [PT21], all non-trivial zeroes ρ with $|\operatorname{Im} \rho| \leq 3 \cdot 10^{12}$ were computationally verified to lie on the critical line. The above bounds represent the current largest known zero-free regions in the range $t > 3 \cdot 10^{12}$. Of these, Theorem 1.1 is sharpest in the interval $3 \cdot 10^{12} < t < \exp(56.5)$, Theorem 1.2 is sharpest for $\exp(56.6) < t < \exp(5.4 \cdot 10^5)$ and Theorem 1.3 is sharpest for $t > \exp(5.5 \cdot 10^5)$.

1.4 Other approaches and conditional results

There exist other (often creative) methods of proving zero-free regions, although these approaches have not yet been applied in an explicit context. We conclude this chapter by briefly reviewing some of these methods, as well as recording some stronger zero-free regions that are possible assuming certain hypotheses.

It is well-known that zero-free regions of $\zeta(s)$ imply certain error estimates in the prime number theorem. Turán [THP84, Theorem 40.1]² showed that the following converse relationship also holds: if one has $\pi(x) - \text{li}(x) \ll x \exp(-A(\log x)^\theta)$ for some fixed $\theta \in (1/2, 1)$, then there are no zeroes in the region $\sigma > 1 - A/(\log t)^{1/\theta-1}$ for t sufficiently large. This provides a way of obtaining zero-free regions using an elementary proof of the prime number theorem (first established by Selberg [Sel49] and Erdős [Erd49]). Presently, the sharpest known prime number estimate obtained via elementary means is $\pi(x) - \text{li}(x) \ll x \exp(-c(\log x)^{1/6})$ for a constant $c > 0$, proved by Srinivasan–Sampath [SS88]. This implies a zero-free region of the form $\sigma > 1 - A/(\log t)^5$ if t is sufficiently large. Although this region is smaller than the largest known zero-free region, the method is interesting since it requires only arithmetic information about the primes.

The approaches reviewed in previous sections all rely on the existence of a non-negative trigonometric polynomial whose coefficients satisfy $a_1 > a_0$. In 1976, Balasubramanian–Ramachandra [BR76] developed a method of analysis with no such dependence, relying instead on the polynomial $1 + \cos \theta$ and the estimate

$$\sum_{X < p \leq 2X} \frac{\log p}{p^\sigma} (1 + \cos(t \log p)) \gg X^{1-\sigma} \quad (1 \ll t \ll X^{1/2})$$

as $X \rightarrow \infty$, obtained via sieve-theoretic methods. Their approach also requires no knowledge about the behaviour of $\zeta(s)$ at height kt ($k \geq 2$). As discussed in [Tit86, page 68], this method is capable of producing a zero-free region of classical strength, i.e. $\sigma > 1 - A/\log t$. Another unrelated sieve-theoretic approach due to Motohashi [Mot78] used the Selberg sieve to pass from bounds on the size of $\zeta(s)$ to a zero-free region, thereby providing an

²A more precise form of this relationship was later obtained by Pintz [Pin80].

alternative approach to Landau’s method of detecting zeroes. When combined with (1.13), Motohashi recovered the Vinogradov–Korobov zero-free region.

There is another loosely-related group of ideas that involve analyzing the interactions between two or more complex zeroes of $\zeta(s)$. Levinson [Lev69] first showed that if the non-trivial zeroes of $\zeta(s)$ are well-spaced close to the line $\sigma = 1$, then there are no zeroes in the region

$$\sigma > 1 - \frac{A}{\log \log t} \quad (1.15)$$

for some constant $A > 0$. Here, “well-spaced” means there is a fixed $\delta > 0$ such that, if ρ, ρ' are zeroes with $\operatorname{Re} \rho, \operatorname{Re} \rho' \geq 1 - \delta$, then $\operatorname{Im}(\rho - \rho') \geq \delta$. This spacing condition certainly holds on average (for δ sufficiently small), since the number of zeroes in the rectangle $[1 - \delta, 1] \times [0, T]$ is known to be $O(T^{2\delta})$ (see e.g. [Bou00]). However, local zero-density estimates are presently not strong enough to ensure this spacing condition *always* holds, and thus (1.15) remains a conditional result. One might then ask what types of zero-free regions are possible given what we currently know about the local density of zeroes. In this direction, Montgomery [Mon71, Chapter 11] showed that if $\rho_0 = \beta_0 + it$ is a zero with $\beta_0 > 1/2$, $1 - \beta_0 \leq \delta \leq 1$ and $t > 0$, then

$$\delta^2 \int_0^1 \int_0^\infty \frac{n(t, w, h) + n(2t, w, h)}{(h + \delta)^5 e^{w/\delta}} dh dw \gg (1 - \beta_0)^{-1}, \quad (1.16)$$

where $n(t, h, w)$ is the number of zeroes in the rectangle $[1 - w, 1] \times [t - h/2, t + h/2]$. Here, the left side represents a weighted average of the number of zeroes close to $1 + it$ and $1 + 2it$. Bounding the left side with the best available zero-density estimates (obtained via Jensen’s formula and (1.13)), Montgomery recovered the Vinogradov–Korobov zero-free region. Later, Ramachandra [Ram78] obtained a pointwise version of (1.16), where the left side is replaced with a sum over the zeroes close to $1 + it$ and $1 + 2it$. A further refinement in Balasubramanian–Ramachandra [BR82] showed that one may ignore the zeroes near $1 + 2it$ at the cost of a small amount of averaging at height t . One consequence of their result is

$$n(t, \delta, \delta) e^{-\delta/X} + \frac{1}{X} \int_{-\delta/2}^{\delta/2} \int_0^{\delta+X} n(t + v, u, u) e^{-u/X} du dv \gg \delta^6 X (1 - \beta_0)^{-1}$$

for sufficiently small $\delta > 0$ and all $\delta^{-4}(1 - \beta_0) \ll X \ll (\log \log t)^{-1}$. Each of these devices can be used in conjunction with (1.13) to recover the Vinogradov–Korobov zero-free region. While these methods currently have no explicit counterparts, it appears possible (at least in principle) that some combination of these ideas can be used to further enlarge the zero-free region.

Chapter 2

The classical zero-free region

2.1 Introduction

In the previous chapter, we reviewed two ideas that were successfully used to enlarge the classical zero-free region of $\zeta(s)$. The first idea, due to Stechkin [Ste70], was to make use of the fact that the complex zeroes of the zeta-function exist in pairs about the critical line. The second idea, due to Heath-Brown [HB92a], was to smooth the classical zero-detector using a weight function f . Recall that the zero-detectors they used have the Dirichlet series representations

$$\sum_{n \geq 1} \frac{\Lambda(n)}{n^s} \left(1 - \frac{\kappa}{n^\delta}\right) \quad \text{and} \quad \sum_{n \geq 1} \frac{\Lambda(n)}{n^s} f(\log n)$$

respectively, where f is a compactly supported, twice-differentiable real-valued function that decays steadily to 0.

Remarkably, the two ideas appear to be largely orthogonal, despite their both involving weighting the classical zero-detector $\sum_{n \geq 1} \Lambda(n)n^{-s}$ by a function $f(\log n)$ (with $f(x) = 1 - \kappa e^{-\delta x}$ in Stechkin's case). This is because the weight functions are used in two different ways. In Stechkin's approach, the weight function allows one to reduce the key quantity $f(0)$ from 1 in the classical case to $1 - \kappa$. Normally, doing so invalidates an important

positivity condition (appearing as (1.5) in the classical proof); however, this problem was avoided by exploiting the existence of zeroes in pairs. Meanwhile, Heath-Brown's approach makes no critical use of the fact that zeroes exist in pairs; instead, the improvement comes from using f to attenuate the zero-detector's tail oscillations so that one may move s inside the critical strip. In terms of shape, the two weight functions are polar opposites: Stechkin's function attenuates terms in the zero-detector corresponding to small n , while Heath-Brown's function attenuates terms corresponding to large n .

One may therefore wonder if the two ideas can be used concurrently. Indeed, Kadiri [Kad05] showed that this can be done profitably by using the zero detector

$$\sum_{n \geq 1} \frac{\Lambda(n)}{n^s} f(\log n) \left(1 - \frac{\kappa'}{n^{\delta'}}\right) \quad (2.1)$$

for some constants $\kappa', \delta' > 0$, and where f is the same test function considered by Heath-Brown. Although the paper contains several other innovations, the combination of smoothing functions explains most of the improvement from $A = (8.463)^{-1}$ [For02b] to $A = (5.69693)^{-1}$ [Kad05].

The modest goal of this chapter is to explore how the two ideas can be combined more efficiently. Our main result is:

Lemma 2.1. *If $2 \leq t \leq \exp(56.6)$ then $\zeta(\sigma + it) \neq 0$ in the region*

$$\sigma > 1 - \frac{1}{4.862 \log t}.$$

Theorem 1.1 then follows from combining this with Lemma 3.1 in the next chapter (since Lemma 3.1 implies a result of the same strength for $t \geq \exp(56.6)$). Our improvement comes from generalizing the “Stechkin factor” $1 - \kappa'/n^{\delta'}$ into a function $h(\log n)$, then choosing h appropriately using a numerical approach.

2.2 Outline of argument

In this section we will roughly describe the argument, and use this to motivate our choice of a smoothing function f . Throughout, we draw heavy inspiration both from the classical proof in the previous chapter and from the argument of Heath-Brown [HB92a]. We also take this opportunity to state plainly a number of observations that are implicit in several works in the literature. First, since non-trivial zeroes are symmetric about the critical line and the real axis, we will just focus on the quadrant $\{z \in \mathbb{C} : \operatorname{Re} z \geq 1/2, \operatorname{Im} z \geq 0\}$. In fact, we can further restrict our focus to

$$\operatorname{Im} z \geq H := 3 \cdot 10^{12} \tag{2.2}$$

since all zeroes $\rho = \beta + i\gamma$ with $0 < \gamma \leq H$ have been computationally verified to lie on the critical line [PT21]. The non-trivial zeroes at heights just above H pose the greatest difficulty — indeed, asymptotically wider zero-free regions are possible at sufficiently large heights. Therefore, we optimize our arguments for zeroes whose imaginary part lies in a finite interval, which explains the form of Lemma 2.1.

Suppose f is a non-negative, bounded and compactly supported function with a well-defined Laplace transform

$$F(z) := \int_0^\infty e^{-zu} f(u) du.$$

Under suitable assumptions, the associated zero-detector has an “explicit formula” of the form

$$\begin{aligned} \operatorname{Re} \sum_{n \geq 1} \frac{\Lambda(n)}{n^s} f(\log n) &= \frac{f(0)}{2} \operatorname{Re} \frac{\Gamma'}{\Gamma} \left(\frac{s}{2} + 1 \right) + \operatorname{Re} F(s-1) \\ &\quad - \sum_{\rho} \operatorname{Re} F(s-\rho) + o(1), \end{aligned} \tag{2.3}$$

where $s = \sigma + it$ and the error term is taken with respect to $t \rightarrow \infty$. This formula may be viewed as a natural generalization of Hadamard’s product formula used in the classical proof, with an important distinction that the formula remains valid for s *inside* the critical strip.

It turns out that moving s inside the critical strip is accretive to the size of the zero-free region, provided that s itself resides in a zero-free region. This inspires an interesting inductive argument: one begins with a known zero-free region, then iteratively¹ expands it by taking s progressively deeper inside the critical strip as permitted by the new zero-free region at each stage. This idea was introduced in [Kad05] and has since become standard in the literature.

Specifically, suppose that there exists a simple², complex zero $\rho_0 = \beta_0 + it$ lying just outside a known zero-free region $\sigma > 1 - A/\log t$ for some constant $A > 0$. We seek to show that ρ_0 actually satisfies a stronger inequality, of the form $\beta_0 \leq 1 - (A + \varepsilon)/\log t$ say, for some $\varepsilon > 0$. Assume for a contradiction that

$$A \leq (1 - \beta_0) \log t < A + \varepsilon. \quad (2.4)$$

We start by considering a non-negative trigonometric polynomial of the form

$$P(\theta) = \sum_{0 \leq k \leq K} a_k \cos(k\theta), \quad (2.5)$$

satisfying $P(\theta) \geq 0$, $a_k > 0$ and $a_1 > a_0$, which generalizes the polynomial $3 + 4 \cos \theta + \cos 2\theta$ used in the classical proof. The existence and construction of such polynomials is non-trivial, and we refer the reader to [MT14] for an account. Combined with (2.3) evaluated at $s = s_k := \sigma + ikt$ for $0 \leq k \leq K$, we obtain

$$\begin{aligned} 0 &\leq \sum_{n \geq 1} \frac{\Lambda(n)}{n^\sigma} f(\log n) P(t \log n) = \operatorname{Re} \sum_{0 \leq k \leq K} a_k \sum_{n \geq 1} \frac{\Lambda(n)}{n^{s_k}} f(\log n) \\ &= \operatorname{Re} \sum_{0 \leq k \leq K} a_k \left(\frac{f(0)}{2} \frac{\Gamma'}{\Gamma} \left(\frac{s_k}{2} + 1 \right) + F(s_k - 1) - \sum_{\rho} F(s_k - \rho) \right) + o(1). \end{aligned} \quad (2.6)$$

Using classical estimates, as $t \rightarrow \infty$,

$$\operatorname{Re} \sum_{0 \leq k \leq K} a_k \left(\frac{f(0)}{2} \frac{\Gamma'}{\Gamma} \left(\frac{s_k}{2} + 1 \right) + F(s_k - 1) \right) < cf(0) \log t + a_0 F(\sigma - 1),$$

¹As usual, there is no free lunch — eventually the marginal progress made with each iteration tends to 0 and the size of the zero-free region stabilizes.

²Since all zeroes in the sum (2.3) are counted with multiplicity, if ρ_0 has multiplicity $m > 1$ then it is included in the sum m times, and we arbitrarily rename one of them ρ_0 . The remaining $m - 1$ terms are treated the same way as the other zeroes $\rho \neq \rho_0$.

for an absolute constant $c > 0$. To bound the sum over zeroes we require an additional assumption on f . Specifically, we assume the following positivity condition: for any non-trivial zero ρ ,

$$\operatorname{Re} F(s - \rho) + \operatorname{Re} F(s - (1 - \bar{\rho})) \geq 0. \quad (2.7)$$

Note that both ρ and $1 - \bar{\rho}$ are (possibly distinct) complex zeroes of zeta, so (2.7) may be viewed as a slightly “averaged” version of the pointwise condition $\operatorname{Re} F(s - \rho) \geq 0$ considered in Heath-Brown [HB92a]. Condition (2.7) is also satisfied by Kadiri’s [Kad05] smoothing function, and it is at this point where the ideas of Stechkin and Heath-Brown are combined. This condition allows us to isolate the contribution of just one pair of zeroes, say $\beta_0 + it$ and $1 - \beta_0 + it$, since

$$\operatorname{Re} \sum_{\rho} F(s_1 - \rho) \geq F(\sigma - \beta_0) + F(\sigma - 1 + \beta_0)$$

and

$$\operatorname{Re} \sum_{\rho} F(s_k - \rho) \geq 0 \quad (k \neq 1).$$

The $F(\sigma - 1 + \beta_0)$ term is $o(1)$, so combining these estimates with (2.6), one has

$$c \log t > \frac{a_1 F(\sigma - \beta_0) - a_0 F(\sigma - 1) + o(1)}{f(0)}. \quad (2.8)$$

Next we choose σ and f so that the right side is $\gg 1/\eta$, where

$$\eta := 1 - \beta_0.$$

This can be achieved, for instance, if one takes $\sigma = 1 - (A + o(1))/\log t$ and $f(u) = \eta g(\eta u)$ for some fixed function g . With these choices the right side of (2.8) is

$$\frac{a_1 G(1 - \mu) - a_0 G(-\mu)}{g(0)} \eta^{-1}$$

where G is the Laplace transform of g and $\mu = (1 - \sigma)/\eta = 1 + O(\varepsilon)$. Taking $\varepsilon \rightarrow 0$, the coefficient of η^{-1} is a constant, as required. Therefore, one arrives at a contradiction to (2.4) if A is small enough, which completes the argument.

2.2.1 An optimization problem

Let us now consider the problem of choosing the best f for each $\eta > 0$ (we now regard η as fixed). First, we review the constraints one needs to place on f . To control the various $o(1)$ error terms we require $F(z) = f(0)/z + O(|z|^{-2})$ as $|z| \rightarrow \infty$. This condition is surprisingly mild³ — in [HB92a] this was enforced by requiring f to be compactly supported and having a bounded and continuous second derivative. The second, more important condition comes from (2.7) — for this it suffices to require that $\operatorname{Re} F(z) + \operatorname{Re} F(2\sigma - 1 - \bar{z}) \geq 0$ for all $\operatorname{Re} z \geq 0$.

Taking $\varepsilon \rightarrow 0$ and ignoring $o(1)$ terms, from (2.8) the optimization problem we are led to consider is (for each fixed η)

$$\max_f \frac{a_1 F(0) - a_0 F(-\eta)}{f(0)} \quad (2.9)$$

subject to $F(z) = f(0)/z + O(|z|^{-2})$ and

$$\operatorname{Re} F(z) + \operatorname{Re} F(2\sigma - 1 - \bar{z}) \geq 0 \quad (\operatorname{Re} z \geq 0). \quad (2.10)$$

In [HB92a], Heath-Brown studied a similar optimization problem, which has the constraint $\operatorname{Re} F(z) \geq 0$ in place of (2.10). In the next section we explore how solutions to Heath-Brown's optimization problem can be adapted to solve (2.9).

2.3 The test function f

In this section we choose the test function f . We emphasize that some of our arguments in this section are not completely rigorous, and in particular we make no claims of optimality. Rather, our goal is to heuristically motivate our choice of f .

First, suppose that w is a twice-differentiable, compactly-supported, real-valued function and let $W(z)$ denote its Laplace transform, which satisfies

$$\operatorname{Re} W(z) \geq 0 \quad (\operatorname{Re} z \geq 0).$$

³In fact, our eventual choice of f satisfies the stronger condition $F(z) = f(0)/z + O(|z|^{-3})$. The improved error term was also noted in [Kad05].

Examples of functions satisfying these conditions are studied by Heath-Brown [HB92a]. One function which is particularly relevant to our application is [HB92a, Lemma 7.1] with $\lambda = 1$, given by:

Definition 2.1 (The functions w and W). *For fixed $\theta \in (0, \pi/2)$, let w be defined as*

$$w(u) := \sec^2 \theta (\sec^2 \theta (\theta \cot \theta - u/2) \cos(u \tan \theta) + 2\theta \cot \theta - u \\ + \sin(2\theta - u \tan \theta) \csc 2\theta - 2(1 + \sin(\theta - u \tan \theta) \csc \theta))$$

for $0 \leq u \leq 2\theta \cot \theta$, and $w(u) := 0$ elsewhere. Furthermore, let $W(z) := \int_0^\infty e^{-zu} w(u) du$ denote its Laplace transform.

In our notation, one choice of f considered by Heath-Brown was

$$\eta w(\eta u). \tag{2.11}$$

In [HB92a], it was shown via a calculus-of-variations argument that this choice of f solves, within a large family of functions, the optimization problem

$$\max_f \frac{a_1 F(0) - a_0 F(-\eta)}{f(0)} \quad \text{subject to} \quad \operatorname{Re} F(z) \geq 0 \quad (\operatorname{Re} z \geq 0).$$

Consider now the function

$$f(u) = \eta w(\eta u) h(u), \quad h(u) := \frac{1}{1 + e^{-(2\sigma-1)u}}. \tag{2.12}$$

This choice of f satisfies (2.10), since for any $z = iy$ ($y \in \mathbb{R}$),

$$\operatorname{Re} F(z) + \operatorname{Re} F(2\sigma - 1 - \bar{z}) = \operatorname{Re} \int_0^\infty e^{-iyu} (1 + e^{-(2\sigma-1)u}) f(u) du \tag{2.13}$$

$$= \operatorname{Re} \int_0^\infty e^{-iyu/\eta} w(u) du \tag{2.14}$$

$$= \operatorname{Re} W(iy/\eta) \geq 0.$$

We can then extend this inequality to hold for all $\operatorname{Re} z \geq 0$ via a theorem of Heath-Brown [HB92a, Lemma 4.1]. In particular, one might expect that this choice of weight function h is uniquely efficient because there is no “loss” in passing from (2.13) to (2.14).

Unfortunately, f does not satisfy the condition $F(z) = f(0)/z + O(|z|^{-2})$. Furthermore, the inverse Laplace transform of h does not converge, which makes analysis difficult. For these reasons, we will instead use a discrete approximation of $h(u)$ which produces numerically similar results, and which possesses the desired properties. Roughly speaking, we choose parameters κ_m ($1 \leq m \leq M$) so that uniformly for $x \in [0, 1]$, one has

$$\sum_{0 \leq m \leq M} \kappa_m x^m \approx \frac{1}{1+x}. \quad (2.15)$$

One may be inclined to find the coefficients κ_m in (2.15) using a Taylor expansion at say $x = 1/2$; however, this leads to approximations that are catastrophically bad at e.g. $x = 0$. Instead, we search for favourable coefficients computationally. As a first pass we choose κ_m to minimize the L^∞ norm

$$\max_{0 \leq x \leq 1} \left| \frac{1}{1+x} - \sum_{0 \leq m \leq M} \kappa_m x^m \right|.$$

Then, we perturb the coefficients so as to make the arguments that follow simpler (in particular, we opt for slightly less favourable coefficients that can be more easily analyzed, and whose rational approximations are “simple” to aid our presentation). In the end we take $M = 6$ and

$$[\kappa_m]_{0 \leq m \leq M} = \left[1, -\frac{851}{859}, \frac{780}{859}, -\frac{525}{859}, \frac{171}{859}, \frac{28}{859}, -\frac{29}{859} \right]. \quad (2.16)$$

Our choice of f is then given by

Definition 2.2 (The functions f and F). *For fixed $\eta > 0$ and $1/2 < \sigma < 1$, define*

$$f(u) := \eta w(\eta u) \sum_{0 \leq m \leq M} \kappa_m e^{-(2\sigma-1)mu} \quad (2.17)$$

where M and κ_m are fixed parameters defined in (2.16). Furthermore, let $F(z) := \int_0^\infty e^{-zu} f(u) du$ denote its Laplace transform.

In comparison, Kadiri’s [Kad05] test function (2.1) may be expressed in our notation as

$$\eta w(\eta u) \sum_{0 \leq m \leq 1} \kappa_m e^{-\delta_m u} \quad (2.18)$$

with $\kappa_0 = 1$, $\delta_0 = 0$, $\kappa_1 = \kappa'$ and $\delta_1 = \delta'$. One can measure the “efficiency” of the test functions (2.11), (2.12), (2.17) and (2.18) by evaluating the functional in (2.9), which in each case gives

$$\frac{a_1 F(0) - a_0 F(-\eta)}{f(0)} = (c + O(\eta)) \frac{a_1 W(0) - a_0 W(-1)}{\eta w(0)},$$

where $c = 859/433 = 1.98\dots$ in the case of (2.17), $c = 2$ for the hypothetical function (2.12), $c = 1.78\dots$ for (2.18) and $c = 1$ for (2.11). In this sense, the value attained by our discrete approximation is within one percent of the hypothetical limit of the method.

Figure 2.1 provides a comparison of the respective test functions.

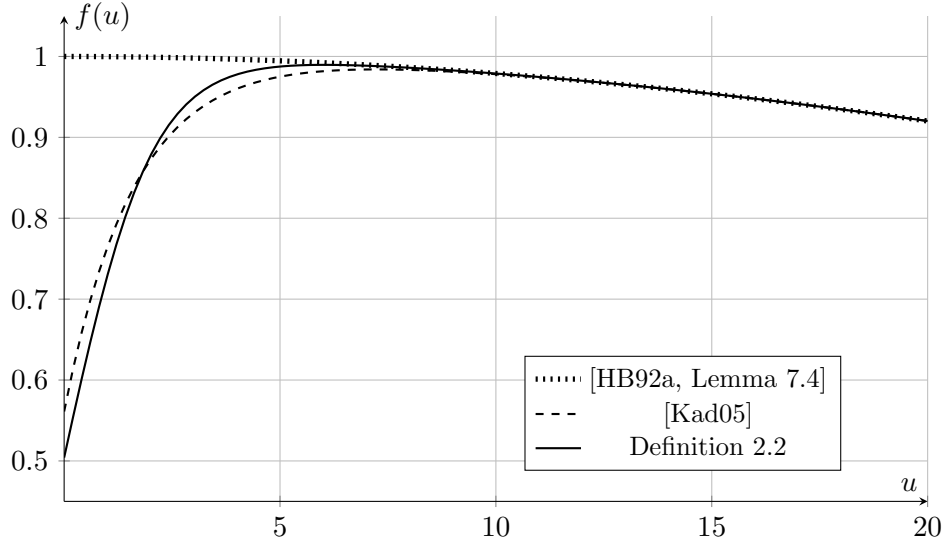


Figure 2.1: Plot of different choices of test functions close to $u = 0$ (normalized so that $\eta w(0) = 1$) when $\sigma = \sigma_0$.

2.3.1 Properties of the functions w and f

Let us now record some properties of these functions, starting with:

Lemma 2.2. *One has $f(u) \geq 0$ for all $u \geq 0$.*

Proof. By inspection $w(u) \geq 0$, so upon taking $x = e^{-(2\sigma-1)u}$ (and noting that the expo-

ment is negative from the assumption $\sigma > 1/2$), it suffices to show that

$$p(x) = \sum_{0 \leq m \leq M} \kappa_m x^m > 0 \quad (0 < x \leq 1),$$

which we verify by numerically isolating all M roots of the polynomial p , checking that none reside on $(0, 1]$, and noting that $p(1) > 0$. $\square$

Next, we record an explicit estimate for $W(z)$, due to Ford [For02b; For22].

Lemma 2.3 (Ford [For02b] §7). *Let $\theta \in (0, \pi/2)$ and $r > \tan \theta$ be fixed, and let W be defined in Definition 2.1. If we write*

$$W(z) = \frac{w(0)}{z} + W_0(z),$$

then for all $\operatorname{Re} z \geq \nu$ and $|z| \geq r$ one has $|W_0(z)| \leq C(\nu, r)|z|^{-3}$, where

$$C(\nu, r) := c_0 r \frac{c_2(r+1)^2(e^{-2\nu\theta \cot \theta} + 1) + c_1 r + c_3 r^3}{(r^2 - \tan^2 \theta)^2}$$

and

$$\begin{aligned} c_0 &= \csc \theta \sec^2 \theta, & c_1 &= (\theta - \sin \theta \cos \theta) \tan^4 \theta, \\ c_2 &= \tan^3 \theta \sin^2 \theta, & c_3 &= (\theta - \sin \theta \cos \theta) \tan^2 \theta. \end{aligned}$$

Before proceeding further we fix

$$\theta = 1.1338 \tag{2.19}$$

in Definition 2.1 and 2.2, the choice of which was determined via numerical experimentation. Thus f and F are now pure functions, and also

$$w(0) = \sec^2 \theta (\theta \tan \theta + 3\theta \cot \theta - 3) = 5.672787598 \dots$$

Recall H is defined in (2.2). For future use we also define the following parameters

$$\begin{aligned} A_0 &= (4.862)^{-1} = 0.2056766762 \dots, \\ T_0 &= 10^{10}, \\ \delta_m &= (2\sigma - 1)m, \\ \eta_0 &= A_0 / \log H = 0.0071590428 \dots, \\ K &= 16, \\ \sigma_0 &= 1 - A_0 / \log(KH + T_0) = 0.9934710854 \dots \end{aligned} \tag{2.20}$$

The parameter A_0 is the constant in the zero-free region of Lemma 2.1. We defined σ_0 so that our eventual choice of σ satisfies $\sigma \in [\sigma_0, 1)$, and similarly one is guaranteed that $\eta \in (0, \eta_0]$. The parameter K is the degree of the non-negative trigonometric polynomial which we discuss in §2.4.

Recall that $\operatorname{Re} W(z) \geq 0$ for $\operatorname{Re} z \geq 0$ (see [HB92a]). This property is used to show that f satisfies the non-negativity condition (2.7).

Lemma 2.4 (Non-negativity property). *Let $\sigma \in [\sigma_0, 1)$, $\eta \in (0, \eta_0]$ and $F(z)$ be as defined in Definition 2.2. Then for all complex numbers z with $0 \leq \operatorname{Re} z \leq 2\sigma - 1$, one has*

$$\operatorname{Re} F(z) + \operatorname{Re} F(2\sigma - 1 - \bar{z}) \geq 0. \quad (2.21)$$

Furthermore, for all $-\eta \leq \operatorname{Re} z \leq 1$ and $|\operatorname{Im} z| \geq T_0$, one has

$$\operatorname{Re} F(z) + \operatorname{Re} F(2\sigma - 1 - \bar{z}) > -\frac{44}{|\operatorname{Im} z|^2} \eta. \quad (2.22)$$

Proof. Since $\operatorname{Re} F(2\sigma - 1 - \bar{z}) = \operatorname{Re} F(2\sigma - 1 - z)$, we may show (2.21) by proving $\operatorname{Re} F(z) + \operatorname{Re} F(2\sigma - 1 - z) \geq 0$ on the boundary of the rectangle $\{z \in \mathbb{C} : 0 \leq \operatorname{Re} z \leq 2\sigma - 1, |\operatorname{Im} z| \leq T\}$ for every sufficiently large T . The result then holds by the maximum modulus principle (applied to the holomorphic function $e^{-F(z) - F(2\sigma - 1 - z)}$).

First we will show that the desired inequality holds on the line $\operatorname{Re} z = 0$ (so that by symmetry it also holds for $\operatorname{Re} z = 2\sigma - 1$). Let us write

$$(1+x) \sum_{0 \leq m \leq M} \kappa_m x^m = \sum_{0 \leq m \leq M+1} b_m x^m.$$

Then

$$\begin{aligned} & \operatorname{Re} F(iy) + \operatorname{Re} F(2\sigma - 1 - iy) \\ &= \operatorname{Re} \int_0^\infty e^{-iyu} (1 + e^{-(2\sigma-1)u}) \eta w(\eta u) \sum_{0 \leq m \leq M} \kappa_m e^{-\delta_m u} du \\ &= \operatorname{Re} \sum_{0 \leq m \leq M+1} b_m \int_0^\infty e^{-(\delta_m + iy)u/\eta} w(u) du \\ &\geq \operatorname{Re} \sum_{1 \leq m \leq M+1} b_m W\left(\frac{\delta_m + iy}{\eta}\right), \end{aligned} \quad (2.23)$$

where we dropped the term corresponding to $m = 0$ since $\operatorname{Re} W(iy/\eta) \geq 0$. If $z = (\delta_m + iy)/\eta$ with $m \geq 1$, then $|z| \geq \operatorname{Re} z \geq (2\sigma_0 - 1)/\eta_0 > 137$. Thus, applying Lemma 2.3,

$$\frac{1}{(2\sigma - 1)w(0)} \left| W_0 \left(\frac{\delta_m + iy}{\eta} \right) \right| \leq \frac{\eta_0^2 C(137, 137)}{(2\sigma_0 - 1)^2 w(0)} \frac{1}{m} \frac{\eta}{|\delta_m + iy|^2} < \frac{\varepsilon_0}{m} \frac{\eta}{|\delta_m + iy|^2}$$

with $\varepsilon_0 = 1/2000$, and also

$$\frac{1}{(2\sigma - 1)w(0)} \operatorname{Re} \frac{w(0)}{(\delta_m + iy)/\eta} = m \frac{\eta}{|\delta_m + iy|^2}.$$

Therefore,

$$\begin{aligned} & \frac{1}{(2\sigma - 1)w(0)\eta} \operatorname{Re} \sum_{1 \leq m \leq M+1} b_m W \left(\frac{\delta_m + iy}{\eta} \right) \\ & \geq \sum_{1 \leq m \leq M+1} \frac{b_m m - |b_m| \varepsilon_0 / m}{|\delta_m + iy|^2} \geq B(y) := \sum_{1 \leq m \leq M+1} b_m B_m(y) \end{aligned}$$

where, since $c_0 m < (2\sigma_0 - 1)m \leq \delta_m \leq m$, where $c_0 = 151/153$,

$$B_m(y) := \begin{cases} \frac{m + \varepsilon_0/m}{c_0^2 m^2 + y^2}, & b_m \leq 0, \\ \frac{m - \varepsilon_0/m}{m^2 + y^2}, & b_m > 0. \end{cases}$$

It suffices to verify that $B(y) \geq 0$ for all $y \in \mathbb{R}$. Substituting the values of κ_m ($0 \leq m \leq M$), one may explicitly derive (with the aid of computer assistance) that $B(y) = p(y)/q(y)$, where

$$\begin{aligned} p(y) = & 4061245152630328137981y^{12} + 4077560173170236734684710y^{10} \\ & - 104378137212291977844887868y^8 - 4484512641017853031179075270y^6 \\ & + 135673322742635307737680349343y^4 - 229732179325278720034298507440y^2 \\ & + 112359769561546903428467326544, \end{aligned}$$

$$\begin{aligned} q(y) = & 4012454647232553285540000(y^2 + 1)(y^2 + 9)(y^2 + 25) \\ & \times (y^2 + 4c_0^2)(y^2 + 16c_0^2)(y^2 + 36c_0^2)(y^2 + 49c_0^2). \end{aligned}$$

By inspection, $q(y)$ has no real roots, and by computationally isolating all twelve complex roots of $p(y)$, we verify that none are real. Therefore, $B(y)$ is one-signed on $\mathbb{R}$ which

together with $B(0) > 0$ implies $B(y) > 0$ for all $y \in \mathbb{R}$. Combined with (2.23), one obtains (2.21) on $\operatorname{Re} z = 0$.

Next we show (2.21) holds on the line segment $\operatorname{Im} z = T$ ($0 \leq \operatorname{Re} z \leq 2\sigma - 1$) for sufficiently large T (and hence also on $\operatorname{Im} z = -T$ since $\operatorname{Re} F(z) = \operatorname{Re} F(\bar{z})$). For this we simply note that for any fixed η and uniformly for $0 \leq x \leq 2\sigma - 1$,

$$\operatorname{Re} F(x + iT) = \operatorname{Re} \sum_{0 \leq m \leq M} \kappa_m W\left(\frac{x + iT}{\eta}\right) = \frac{\eta w(0)}{T^2}(\kappa + o(1)) \quad (T \rightarrow \infty),$$

where $\kappa := \sum_{0 \leq m \leq M} \kappa_m > 0$, so $\operatorname{Re} F(x + iT) > 0$ for sufficiently large T . Thus (2.21) holds.

Now consider (2.22). In light of the bound (2.21) it suffices to show (2.22) for $-\eta \leq \operatorname{Re} z < 0$ and $\operatorname{Im} z \geq T_0$. First, suppose $u = x + iy$ is a complex number and v a real number, such that $|v| \leq \eta$, $x \geq 0$ and $|y| \geq T_0$. Then, by Lemma 2.3 one has

$$W\left(\frac{u+v}{\eta}\right) - W\left(\frac{u}{\eta}\right) = \eta w(0) \left(\frac{1}{u+v} - \frac{1}{u}\right) + W_0\left(\frac{u+v}{\eta}\right) - W_0\left(\frac{u}{\eta}\right),$$

where, since $\operatorname{Re}(u/\eta) \geq 0$, $\operatorname{Re}(u+v)/\eta \geq -1$ and $|y/\eta| \geq T_0/\eta_0$ one has

$$\left|W_0\left(\frac{u+v}{\eta}\right)\right| \leq \frac{\eta_0^2 C(-1, T_0/\eta_0)}{T_0} \frac{\eta}{y^2}, \quad \left|W_0\left(\frac{u}{\eta}\right)\right| \leq \frac{\eta_0^2 C(0, T_0/\eta_0)}{T_0} \frac{\eta}{y^2}.$$

Meanwhile,

$$\left|\frac{1}{u+v} - \frac{1}{u}\right| = \frac{|v|}{|u||u+v|} \leq \frac{\eta}{y^2}.$$

Combining everything,

$$\left|W\left(\frac{u+v}{\eta}\right) - W\left(\frac{u}{\eta}\right)\right| \leq \frac{\eta}{y^2} \left(w(0) + \frac{C(-1, T_0/\eta_0) + C(0, T_0/\eta_0)}{T_0} \eta_0^2\right) < \frac{5.7\eta}{y^2}.$$

Taking $u = \delta_m + iy$ ($m \geq 0$) and $v = \operatorname{Re} z$, one gets

$$\operatorname{Re} F(z) = \sum_{0 \leq m \leq M} \kappa_m W\left(\frac{z + \delta_m}{\eta}\right) > \operatorname{Re} F(iy) - \frac{5.7\eta}{y^2} \sum_{0 \leq m \leq M} |\kappa_m|.$$

Similarly, if we instead take $u = 2\sigma - 1 + \delta_m + iy$ and $v = -\operatorname{Re} z$, then

$$\operatorname{Re} F(2\sigma - 1 - \bar{z}) > \operatorname{Re} F(2\sigma - 1 + iy) - \frac{5.7\eta}{y^2} \sum_{0 \leq m \leq M} |\kappa_m|.$$

Adding the two inequalities gives (2.22), since $\operatorname{Re} F(iy) + \operatorname{Re} F(2\sigma - 1 + iy) \geq 0$ by (2.21). $\square$

Lemma 2.4 is used to isolate the contribution of a pair of zeroes at height t , while discarding all other zeroes (incurring a small loss).

Lemma 2.5. *Suppose $\zeta(\beta_0 + it) = 0$ with $\beta_0 > 1/2$ and $t \geq H$. Let $A \in (0, A_0]$, $\eta \in (0, A_0/\log t]$ and F be as defined in Definition 2.2. Furthermore suppose that ζ has no zeroes $\beta + i\gamma$ in the region $\beta \geq 1 - A/\log \gamma$ and $\gamma \geq H$. Let $s_k = \sigma + ikt$ where $0 \leq k \leq K$ and $\sigma = 1 - A/\log(Kt + T_0)$. Then*

$$\operatorname{Re} \sum_{\rho} F(s_1 - \rho) \geq F(\sigma - \beta_0) + F(\sigma - 1 + \beta_0) - 10^{-8}.$$

Furthermore, for any $k \neq 1$ one has

$$\operatorname{Re} \sum_{\rho} F(s_k - \rho) \geq -10^{-8}.$$

Proof. Throughout let us write

$$T = Kt + T_0.$$

We divide the zeroes $\rho = \beta + i\gamma$ of ζ in the critical strip into those with $|\gamma| \leq T$ and those with $|\gamma| > T$.

Suppose first that $|\gamma| \leq T$. Then

$$\beta \leq 1 - A/\log |\gamma| \leq 1 - A/\log T = \sigma$$

so that $\operatorname{Re}(s_k - \rho) \geq 0$ for all such zeroes. A similar argument shows that $\operatorname{Re}(s_k - \rho) \leq 2\sigma - 1$. Also, if ρ is a complex zero off the critical line, then $1 - \bar{\rho}$ is also a zero. On the other hand, if ρ lies on the critical line then $s_k - \rho = s_k - (1 - \bar{\rho})$. Thus

$$\begin{aligned} \operatorname{Re} \sum_{\substack{\rho=\beta+i\gamma \\ |\gamma|\leq T}} F(s_k - \rho) &= \frac{1}{2} \operatorname{Re} \sum_{\substack{\rho=1/2+i\gamma \\ |\gamma|\leq T}} (F(s_k - \rho) + F(s_k - (1 - \bar{\rho}))) \\ &\quad + \operatorname{Re} \sum_{\substack{\rho=\beta+i\gamma \\ \beta>1/2 \\ |\gamma|\leq T}} (F(s_k - \rho) + F(s_k - (1 - \bar{\rho}))). \end{aligned}$$

First, suppose $k \neq 1$. Since $0 \leq \operatorname{Re}(s_k - \rho) \leq 2\sigma - 1$ we may apply (2.21) termwise with $z = s - \rho$, so that both sums on the right side are non-negative. Therefore

$$\operatorname{Re} \sum_{\substack{\rho=\beta+i\gamma \\ |\gamma|\leq T}} F(s_k - \rho) \geq 0 \quad (k \neq 1). \quad (2.24)$$

On the other hand if $k = 1$, then we isolate the term corresponding to $\rho = \rho_0$ in the second sum, and apply (2.21) to all other terms (recall that $\rho_0 = \beta_0 + it$ is our hypothetical zero off the critical line). We obtain

$$\operatorname{Re} \sum_{\substack{\rho=\beta+i\gamma \\ |\gamma|\leq T}} F(s_1 - \rho) \geq F(\sigma - \beta_0) + F(2\sigma - 1 + \beta_0). \quad (2.25)$$

It remains to estimate the contribution of those zeroes with $|\gamma| > T$. These zeroes are far from s so the corresponding terms are small, and we can afford to estimate them crudely. Applying (2.22) termwise, one obtains

$$\begin{aligned} \operatorname{Re} \sum_{\substack{\rho=\beta+i\gamma \\ |\gamma|>T}} F(s_k - \rho) &= \operatorname{Re} \sum_{\substack{\rho=\beta+i\gamma \\ \gamma>T}} (F(s_k - \rho) + F(s_k - \bar{\rho})) \\ &\geq -44\eta \sum_{\substack{\rho=\beta+i\gamma \\ \gamma>T}} \left(\frac{1}{(\gamma - kt)^2} + \frac{1}{(\gamma + kt)^2} \right). \end{aligned}$$

Here we have made use of the fact that $|\operatorname{Im}(s_k - \rho)|, |\operatorname{Im}(s_k - \bar{\rho})| > T_0$ for each zero in the sum. If $k = 0$ then we use [Leh66, Lemma 2] with $n = 2$ to get

$$\sum_{\gamma>T_0} \frac{1}{\gamma^2} < \frac{\log T_0}{T_0}$$

so that, since $\eta \leq A_0/\log t \leq \eta_0$ by assumption,

$$\operatorname{Re} \sum_{\substack{\rho=\beta+i\gamma \\ |\gamma|>T}} F(\sigma - \rho) > -88\eta_0 \sum_{\gamma>T_0} \frac{1}{\gamma^2} > -10^{-8}. \quad (2.26)$$

If $k \neq 0$, then we use the same treatment as [MT14]. Applying [Leh66, Lemma 1] with $\phi(x) = (x - kt)^{-2} + (x + kt)^{-2}$, we get

$$\sum_{\gamma>T} \phi(\gamma) = \frac{1}{2\pi} \int_T^\infty \phi(x) \log \frac{x}{2\pi} dx + O^* \left(4\phi(T) \log T + 2 \int_T^\infty \frac{\phi(x)}{x} dx \right),$$

where, here and henceforth, $O^*(A)$ means a complex number whose modulus does not exceed A . Recall that $T = Kt + T_0$ and $k \leq K$, so that

$$\begin{aligned} \int_T^\infty \phi(x) \log \frac{x}{2\pi} dx &\leq \int_{T_0}^\infty \left(\frac{1}{x^2} + \frac{1}{(x + 2kt)^2} \right) \log \frac{kt + x}{2\pi} dx \\ &= \frac{1}{T_0} \log \frac{kt + T_0}{2\pi} + \frac{1}{2kt + T_0} \log \frac{kt + T_0}{2\pi} + \frac{\log(2kt + T_0)}{kt} - \frac{\log T_0}{kt}. \end{aligned}$$

We drop the last term, and bound the second and third terms using $kt \geq H$, since they are both decreasing in kt . Finally, the first term may be bounded using $\log(1+x) \leq x$, so that

$$\log \frac{kt + T_0}{2\pi} \leq \log \frac{Kt}{2\pi} \left(1 + \frac{T_0}{Kt}\right) \leq \log t + \frac{T_0}{KH} + \log \frac{K}{2\pi}.$$

Computing the constants explicitly, one finds

$$\int_{T_0}^{\infty} \phi(x) \log \frac{x}{2\pi} dx < 10^{-10}(\log t + 1). \quad (2.27)$$

Next, since $kt \geq H$, we have

$$\phi(T_0) \log T_0 \leq \left(\frac{1}{T_0^2} + \frac{1}{(2H + T_0)^2} \right) \log(kt + T_0) < 10^{-10}(\log t + 1), \quad (2.28)$$

$$\int_{T_0}^{\infty} \frac{\phi(x)}{x} dx \leq \int_{T_0}^{\infty} \left(\frac{1}{x^2} + \frac{1}{(2H + x)^2} \right) \frac{dx}{x} < 10^{-10}, \quad (2.29)$$

where in deriving (2.28) we used $\log(kt + T_0) \leq \log t + T_0/(KH) + \log K$. Combining (2.27), (2.28) and (2.29), and since $\eta \leq A_0/\log t \leq \eta_0$ by assumption

$$\operatorname{Re} \sum_{\substack{\rho=\beta+i\gamma \\ |\gamma|>T}} F(s_k - \rho) > -44\eta \left(\frac{10^{-10}}{2\pi} + 6 \cdot 10^{-11} \right) (\log t + 1) > -10^{-8}.$$

The result follows from combining the above inequality with (2.24), (2.25) and (2.26) (with some room to spare). $\square$

One can use the assumed properties of f to obtain an “explicit formula”. First, recall the following theorem from [Kad05].

Lemma 2.6 (Kadiri [Kad05] Proposition 2.1). *Let $d > 0$ and suppose g is a real-valued function supported on $[0, d]$ and twice continuously differentiable on $[0, d]$, satisfying $g(d) = g'(0) = g'(d) = g''(d) = 0$. Let $G(z) := \int_0^\infty e^{-zu} g(u) du$ and $G_2(z) := \int_0^\infty e^{-zu} g''(u) du$ denote respectively the Laplace transforms of g and g'' . Then, for all complex numbers s ,*

$$\begin{aligned} \operatorname{Re} \sum_{n \geq 1} \frac{\Lambda(n)}{n^s} g(\log n) &= g(0) \left(-\frac{1}{2} \log \pi + \frac{1}{2} \operatorname{Re} \frac{\Gamma'}{\Gamma} \left(\frac{s}{2} + 1 \right) \right) + \operatorname{Re} G(s-1) \\ &\quad - \operatorname{Re} \sum_{\rho} G(s-\rho) + \operatorname{Re} \left(\frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \frac{G_2(s-z)}{(s-z)^2} \operatorname{Re} \frac{\Gamma'}{\Gamma} \left(\frac{z}{2} \right) dz + \frac{G_2(s)}{s^2} \right). \end{aligned}$$

The next lemma is used to control the error term in the case relevant to our application.

Lemma 2.7 (Bounding the error term). *Let $s = \sigma + it$, $\sigma \geq \sigma_0$ and $\eta \in (0, \eta_0]$. Assume either $t = 0$ or $t \geq H$ and $\eta \leq A_0/\log t$. If $W_0(z) = W(z) - w(0)/z$ then*

$$\left| \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} W_0\left(\frac{s-z}{\eta}\right) \operatorname{Re} \frac{\Gamma'}{\Gamma}\left(\frac{z}{2}\right) dz + W_0\left(\frac{s}{\eta}\right) \right| \leq \begin{cases} 723\eta^3, & t = 0, \\ 14\eta^2 + 425\eta^3, & t \geq H. \end{cases}$$

Proof. Let $E_1(s)$ denote the left side of the above inequality, so that

$$\begin{aligned} E_1(s) &\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| W_0\left(\frac{s - (1/2 + iy)}{\eta}\right) \operatorname{Re} \frac{\Gamma'}{\Gamma}\left(\frac{1/2 + iy}{2}\right) \right| dy + |W_0(s/\eta)| \\ &= T_1 + T_2, \end{aligned}$$

say. For convenience, throughout we write $x_0 := \sigma_0 - 1/2$. By assumption, $\eta \leq \eta_0$. If $\xi = (s - 1/2 - iy)/\eta$ then $|\xi| \geq \operatorname{Re} \xi \geq x_0/\eta_0$. Applying Lemma 2.3 (and noting that $C(x_0/\eta_0, x_0/\eta_0) < 52$), one has

$$\left| W_0\left(\frac{s - 1/2 - iy}{\eta}\right) \right| \leq \frac{52\eta^3}{(x_0^2 + (t - y)^2)^{3/2}}. \quad (2.30)$$

Meanwhile, as in [Kad05, Lemma 3.6] we use

$$\left| \operatorname{Re} \frac{\Gamma'}{\Gamma}\left(\frac{1/2 + iy}{2}\right) \right| \leq U(y) := \begin{cases} \frac{1}{2} \log \frac{16}{1 + 4y^2} + \frac{2}{1 + 4y^2} + 2, & |y| < 1/2, \\ \left| \log \frac{|y|}{2} - \frac{2}{1 + 4y^2} \right| + \frac{2}{3|y|} + \frac{1}{8y^2}, & |y| \geq 1/2. \end{cases}$$

If $t = 0$ then via direct computation,

$$\int_{-\infty}^{\infty} \frac{U(y)}{(x_0^2 + y^2)^{3/2}} dy < 36. \quad (2.31)$$

On the other hand if $t \geq H$ then we divide the integral as

$$\int_{-\infty}^{\infty} \frac{U(t + y)}{(x_0^2 + y^2)^{3/2}} dy = \int_{-\infty}^{-(t-2)} + \int_{-(t-2)}^{t-2} + \int_{t-2}^{\infty} = I_1 + I_2 + I_3.$$

One may verify that $U(x) < \log(100 - x)$ for $x \leq 2$, so that

$$I_1 < \int_{H-2}^{\infty} \frac{\log(y + 100)}{y^3} dy < 10^{-10}.$$

For $x \geq 2$, one may check that $U(x) \leq \log x$ so

$$I_2 \leq \int_{-(t-2)}^{t-2} \frac{\log(t+y)}{(x_0^2 + y^2)^{3/2}} dy < \log t \int_{|y| \leq t-2} \frac{dy}{(x_0^2 + y^2)^{3/2}} < \frac{2}{x_0^2} \log t$$

and also

$$I_3 \leq \int_{H-2}^{\infty} \frac{\log y}{y^3} dy < 10^{-10}.$$

Combining these estimates with (2.30) and (2.31), and using $\eta \leq A_0/\log t \leq \eta_0$ one has

$$T_1 \leq \frac{52}{2\pi} \eta^3 \int_{-\infty}^{\infty} \frac{U(y)}{(x_0^2 + y^2)^{3/2}} dy < \begin{cases} 298\eta^3, & t = 0, \\ 14\eta^2, & t \geq H. \end{cases} \quad (2.32)$$

Next if $\xi = s/\eta$ then $|\xi| \geq \operatorname{Re} \xi \geq \sigma_0/\eta_0$ so that by Lemma 2.3, for all $k \geq 0$

$$T_2 \leq \frac{C(\sigma_0/\eta_0, \sigma_0/\eta_0)}{x_0^3} \eta^3 < 425\eta^3. \quad (2.33)$$

The desired result follows from combining (2.32) and (2.33). $\square$

Lemma 2.8 (The explicit formula). *Let $s = \sigma + it$ with $\sigma \in [\sigma_0, 1)$ and $t \geq H$, and suppose $\eta \in (0, A_0/\log t]$. If f and F are as in Definition 2.2, then*

$$\begin{aligned} \operatorname{Re} \sum_{n \geq 1} \frac{\Lambda(n)}{n^s} f(\log n) &= f(0) \sum_{0 \leq m \leq M} \kappa_m \left(-\frac{1}{2} \log \pi + \frac{1}{2} \operatorname{Re} \frac{\Gamma'}{\Gamma} \left(\frac{s + \delta_m}{2} + 1 \right) \right) \\ &\quad + \operatorname{Re} F(s-1) - \operatorname{Re} \sum_{\rho} F(s-\rho) + E, \end{aligned}$$

where

$$|E| < \begin{cases} 2730\eta^3, & t = 0, \\ 53\eta^2 + 1605\eta^3, & t > H. \end{cases}$$

Proof. Note that if $g(u) = \eta w(\eta u)$ then $g(d) = g'(0) = g'(d) = g''(d)$ and

$$\frac{G_2(z)}{z^2} = G(z) - \frac{g(0)}{z} = W_0(z/\eta).$$

We apply Lemma 2.8 with this choice of g and with s replaced by $s + \delta_m$ ($0 \leq m \leq M$):

$$\begin{aligned}
 \operatorname{Re} \sum_{n \geq 1} \frac{\Lambda(n)}{n^s} f(\log n) &= \operatorname{Re} \sum_{0 \leq m \leq M} \kappa_m \sum_{n \geq 1} \frac{\Lambda(n)}{n^{s+\delta_m}} \eta w(\eta \log n) \\
 &= \operatorname{Re} \sum_{0 \leq m \leq M} \kappa_m \left(\eta w(0) \left(-\frac{1}{2} \log \pi + \frac{1}{2} \frac{\Gamma'}{\Gamma} \left(\frac{s + \delta_m}{2} + 1 \right) \right) \right. \\
 &\quad \left. + W \left(\frac{s + \delta_m - 1}{\eta} \right) - \sum_{\rho} W \left(\frac{s + \delta_m - \rho}{\eta} \right) + E_1(s + \delta_m) \right) \\
 &= f(0) \operatorname{Re} \sum_{0 \leq m \leq M} \kappa_m \left(-\frac{1}{2} \log \pi + \frac{1}{2} \frac{\Gamma'}{\Gamma} \left(\frac{s + \delta_m}{2} + 1 \right) \right) \\
 &\quad + \operatorname{Re} F(s - 1) - \operatorname{Re} \sum_{\rho} F(s - \rho) + \sum_{0 \leq m \leq M} \kappa_m E_2(s + \delta_m)
 \end{aligned}$$

where

$$E_2(s) := \operatorname{Re} \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} W_0 \left(\frac{s-z}{\eta} \right) \operatorname{Re} \frac{\Gamma'}{\Gamma} \left(\frac{z}{2} \right) dz + \operatorname{Re} W_0(s/\eta).$$

Applying Lemma 2.7 (with s replaced by $s + \delta_m$ ($0 \leq m \leq M$) as required), and using the values of κ_m , one has

$$\sum_{0 \leq m \leq M} |\kappa_m E_2(s + \delta_m)| < \begin{cases} 2730\eta^3, & t = 0, \\ 53\eta^2 + 1605\eta^3, & t > H \end{cases}$$

as required. □

2.4 The trigonometric polynomial P

As we briefly mentioned in §2.2, we require a non-negative trigonometric polynomial

$$P(x) := \sum_{0 \leq k \leq K} a_k \cos kx \geq 0$$

with coefficients satisfying $a_0 < a_1$ and $a_k \geq 0$ ($0 \leq k \leq K$). This is a generalization of the polynomial $2(1 + \cos x)^2 = 3 + 4 \cos x + \cos 2x$ used in the classical proof. The task of finding favourable polynomials was considered in [Kon77] and [MT14]. In this chapter we will use the following trigonometric polynomial, which is a rational approximation to the

polynomial found in [MT14, Table 5]

$$P(x) := \frac{1}{14912370} \left| \sum_{0 \leq k \leq K} c_k e^{ikx} \right|^2$$

where we recall that $K = 16$, $c_0 = 4$ and

$$\begin{array}{llll} c_1 = -8, & c_2 = 2, & c_3 = 20, & c_4 = -9, \\ c_5 = -34, & c_6 = 27, & c_7 = 91, & c_8 = -27, \\ c_9 = -201, & c_{10} = 32, & c_{11} = 895, & c_{12} = 1949, \\ c_{13} = 2389, & c_{14} = 1896, & c_{15} = 949, & c_{16} = 239. \end{array}$$

One may verify via a routine computation that all assumed conditions on $P(x)$ are satisfied.

In particular one has

$$a_0 = 1, \quad a_1 = \frac{865534}{497079}, \quad a := \sum_{1 \leq k \leq K} a_k = \frac{2919857}{828465}. \quad (2.34)$$

In standard proofs of the zero-free region, the non-negativity of $P(x)$ is used to establish the inequality

$$\sum_{n \geq 1} \frac{\Lambda(n)}{n^\sigma} f(\log n) P(t \log n) \geq 0.$$

In the remainder of this section, we show that it is possible to strengthen this inequality by replacing the right side with a positive quantity. This improvement will be used later to obtain a larger zero-free region.

Lemma 2.9. *Let $f(u)$ be defined in Definition 2.2 and $P(x)$ be defined as above. Then*

$$\sum_{n \geq 1} \frac{\Lambda(n)}{n^\sigma} f(\log n) P(t \log n) \geq \frac{1}{12} f(0).$$

Our main idea is that instead of applying the estimate $P(t \log n) \geq 0$ pointwise, we first sum over terms of the form $n = p^m$ ($m \geq 1$) where p is a fixed prime. Here we do not attempt to obtain the sharpest possible bounds; rather, our goal is merely to show that some improvement is possible. In particular we only consider the contributions of the primes 2 and 3 since our method is not sharp enough to extract meaningful contributions from the other primes; however, in principle a sufficiently careful analysis should allow one to consider all primes of size $O(\exp(1/\eta))$.

Lemma 2.10. *If $s_k = \sigma + ikt$ with $\sigma \in [\sigma_0, 1)$ and $t \in \mathbb{R}$, then*

$$\operatorname{Re} \sum_{0 \leq k \leq K} \frac{a_k}{p^{s_k} - 1} > \delta_p := \begin{cases} 0.21, & p = 2, \\ 0.02, & p = 3. \end{cases}$$

Proof. Write $\phi_p = \phi_p(\sigma) := (p^\sigma + p^{-\sigma})/2$ and $x = t \log p$ so that

$$\operatorname{Re} \sum_{0 \leq k \leq K} \frac{a_k}{p^{s_k} - 1} = \frac{p^{2\sigma} - 1}{4p^\sigma} \sum_{0 \leq k \leq K} \frac{a_k}{\phi_p(\sigma) - \cos kx} - \frac{1}{2} \sum_{0 \leq k \leq K} a_k. \quad (2.35)$$

Viewed as a function of x , the expression on the right side has period 2π so it suffices to lower-bound

$$\sum_{0 \leq k \leq K} \frac{a_k}{\phi_p(\sigma) - \cos kx} \quad (-\pi \leq x < \pi).$$

If $y = \cos x$ then $\cos kx = T_k(y)$, where T_k is the k th Chebyshev polynomial of the first kind, the first few of which are

$$\begin{aligned} T_0(y) &= 1, & T_1(y) &= y, & T_2(y) &= 2y^2 - 1, \\ T_3(y) &= 4y^3 - 3y, & T_4(y) &= 8y^4 - 8y^2 + 1. \end{aligned}$$

Therefore, noting that $1 < \phi_p(\sigma) < \phi_p(1)$ so each term in the sum is positive, one has

$$\sum_{0 \leq k \leq K} \frac{a_k}{\phi_p(\sigma) - \cos kx} > \sum_{0 \leq k \leq 4} \frac{a_k}{\phi_p(1) - T_k(y)} + \sum_{5 \leq k \leq K} \frac{a_k}{\phi_p(1) + 1} = h_p(y),$$

say. We compute the minimum of $h_p(y)$ for $y \in [-1, 1]$. In the case of $p = 2$ one has

$$h_2(y) > \frac{16510991}{2485395}$$

since this inequality is equivalent to

$$\begin{aligned} & -2894773829632y^{10} + 1704044564480y^9 + 7702262585344y^8 \\ & - 3358453172736y^7 - 7070063534336y^6 + 1811757646464y^5 \\ & + 2378922223840y^4 - 172357278640y^3 - 109315955656y^2 \\ & + 6202997496y + 2536883955 > 0 \end{aligned}$$

which one may verify by numerically isolating all 10 complex roots and checking that none exist on the interval $[-1, 1]$. Similarly, one may show that $h_3(y) > 8592212/2485395$. Combining this with (2.35), and using $(p^{2\sigma} - 1)/p^\sigma \geq (p^{2\sigma_0} - 1)/p^{\sigma_0}$, one obtains

$$\operatorname{Re} \frac{a_k}{p^{s_k} - 1} > \frac{p^{2\sigma_0} - 1}{4p_0^\sigma} \inf_{|y| \leq 1} h_p(y) - \frac{1}{2} \sum_{0 \leq k \leq K} a_k > \delta_p,$$

as required. $\square$

Lemma 2.11. *If f is as defined in Definition 2.2, then $f(u) \geq \kappa f(0)$ for $0 \leq u \leq 59$, where $\kappa = \sum_{0 \leq m \leq M} \kappa_m$.*

Proof. Note that $w'(x) \leq 0$ for $x \geq 0$, so $w(\eta u) \geq w(\eta_0 u)$. Meanwhile, $2\sigma_0 - 1 \leq \delta_m \leq m$ so

$$\eta^{-1} f(u) = w(\eta u) \sum_{0 \leq m \leq M} \kappa_m e^{-\delta_m u} \geq w(\eta_0 u) \sum_{0 \leq m \leq M} \kappa_m e^{-d_m u},$$

where $d_m = (2\sigma_0 - 1)m$ if $\kappa_m < 0$ and $d_m = m$ otherwise. The right side is a pure function of u , and a routine computer-assisted computation is used to verify that

$$-\kappa w(0) + w(\eta_0 u) \sum_{0 \leq m \leq M} \kappa_m e^{-d_m u} \geq 0$$

for $0 \leq u \leq 59$. The result follows from multiplying by η . $\square$

Using the non-negativity of $f(u)$ and $P(x)$, we drop all terms with $n > N := \exp(59)$.

Together with Lemma 2.11, we have

$$\begin{aligned} \sum_{n \geq 1} \frac{\Lambda(n)}{n^\sigma} f(\log n) P(t \log n) &\geq \kappa f(0) \sum_{1 \leq n \leq N} \frac{\Lambda(n)}{n^\sigma} P(t \log n) \\ &= \kappa f(0) \operatorname{Re} \sum_p \log p \sum_{0 \leq k \leq K} a_k \sum_{1 \leq m \leq \log N / \log p} p^{-ms_k}. \end{aligned}$$

The inner sum can be estimated using

$$\operatorname{Re} \sum_{1 \leq m \leq R} p^{-ms_k} = \operatorname{Re} \frac{1}{p^{s_k} - 1} - \operatorname{Re} \sum_{m > R} p^{-ms_k} \geq \operatorname{Re} \frac{1}{p^{s_k} - 1} - \frac{p^{-(R-1)\sigma_0}}{1 - p^{-\sigma_0}},$$

since $\sigma \geq \sigma_0$. Consider now just the terms with $p = 2$ or 3 . Applying the above inequality with $R = \log N / \log p$ and using Lemma 2.10,

$$\sum_{0 \leq k \leq K} a_k \sum_{1 \leq m \leq \log N / \log p} p^{-ms_k} \geq \delta_p - \frac{1}{N} \frac{p^{\sigma_0}}{1 - p^{-\sigma_0}} \sum_{0 \leq k \leq K} a_k > \delta_p - 10^{-10}.$$

Therefore, dropping all terms with $p > 3$,

$$\sum_{n \geq 1} \frac{\Lambda(n)}{n^\sigma} f(\log n) P(t \log n) \geq \kappa f(0) \sum_{p=2,3} (\delta_p - 10^{-10}) \log p > 0.086 f(0)$$

from which Lemma 2.9 follows.

2.5 Proof of Lemma 2.1

To complete the argument we require a few more results to estimate certain terms appearing on the right side of Lemma 2.8; these are recorded in the next few lemmas.

Lemma 2.12. *Let $s_k = \sigma + ikt$ with $k \geq 0$ an integer, $\sigma_0 \leq \sigma < 1$ and $t \geq H$. If κ_m ($0 \leq m \leq M$) and a_k ($0 \leq k \leq K$) are as defined in (2.16) and Section 2.4 respectively, then*

$$\sum_{0 \leq k \leq K} a_k \sum_{0 \leq m \leq M} \kappa_m \left(-\frac{1}{2} \log \pi + \frac{1}{2} \operatorname{Re} \frac{\Gamma'}{\Gamma} \left(\frac{s_k + \delta_m}{2} + 1 \right) \right) \leq \frac{a\kappa}{2} \log t - 1.568$$

where $a := \sum_{1 \leq k \leq K} a_k = 2919857/828465$, $\kappa := \sum_{0 \leq m \leq M} \kappa_m = 433/859$.

Proof. For $k \geq 1$, we use the following estimate, due to [Kad05, Equation (24)]: if $z = x + iy$ with $0 < x < |y|$, then

$$\operatorname{Re} \frac{\Gamma'}{\Gamma}(z) = \log |y| - \frac{1}{2} \frac{x}{|z|^2} + O^* \left(\frac{1}{12x|y|} + \frac{x^2}{2y^2} \right).$$

Applying this estimate with $z = (s_k + \delta_m)/2 + 1$, all error terms for $k \geq 1$ are $O(t^{-1})$ so we can afford to estimate them roughly. In particular, for $1 \leq x \leq (M+3)/2$ and $y \geq H/2$ one has

$$\left| -\frac{1}{2} \frac{x}{|z|^2} + \frac{1}{12x|y|} + \frac{x^2}{2y^2} \right| < 10^{-10}.$$

Therefore

$$\begin{aligned}
 & \sum_{1 \leq k \leq K} a_k \sum_{0 \leq m \leq M} \kappa_m \left(-\frac{1}{2} \log \pi + \frac{1}{2} \operatorname{Re} \frac{\Gamma'}{\Gamma} \left(\frac{s_k + \delta_m}{2} + 1 \right) \right) \\
 & \leq \frac{\kappa}{2} \sum_{1 \leq k \leq K} a_k \left(-\log \pi + \log \frac{kt}{2} + 10^{-10} \right) \\
 & = \frac{a\kappa}{2} \log t + \frac{a\kappa}{2} (-\log 2\pi + 10^{-10}) + \frac{\kappa}{2} \sum_{1 \leq k \leq K} a_k \log k.
 \end{aligned} \tag{2.36}$$

For the term corresponding to $k = 0$ we use the fact that $\Gamma'/\Gamma(x)$ is increasing for $x > 0$, so

$$\frac{\Gamma'}{\Gamma} \left(\frac{\sigma_0 + (2\sigma_0 - 1)m}{2} + 1 \right) \leq \frac{\Gamma'}{\Gamma} \left(\frac{\sigma + \delta_m}{2} + 1 \right) \leq \frac{\Gamma'}{\Gamma} \left(\frac{m+3}{2} \right).$$

Therefore, by choosing the appropriate bound depending on the sign of κ_m , we find

$$\frac{1}{2} \sum_{0 \leq m \leq M} \kappa_m \frac{\Gamma'}{\Gamma} \left(\frac{\sigma + \delta_m}{2} + 1 \right) \leq -0.041. \tag{2.37}$$

The result follows from combining (2.36) and (2.37). $\square$

Lemma 2.13. *Suppose $s_k = \sigma + ikt$ ($0 \leq k \leq K$) with $\sigma_0 \leq \sigma < 1$ and $t \geq H$. Let $\eta \in (0, \eta_0]$ and F be as defined in Definition 2.2. Then*

$$\operatorname{Re} \sum_{0 \leq k \leq K} a_k F(s_k - 1) \leq a_0 F(\sigma - 1) + 10^{-10} \eta.$$

Proof. Suppose that $k \geq 1$. By Lemma 2.3, for $\operatorname{Re} z \geq \nu, |z| \geq r$, one has

$$\operatorname{Re} W(z) = \frac{w(0) \operatorname{Re} z}{|z|^2} + O^*(C(\nu, r) |z|^{-3}).$$

Taking $z = (s_k + \delta_m - 1)/\eta$, so that $|z| \geq t/\eta \geq H/\eta_0 > 10^{10}$ and $\operatorname{Re} z \geq (\sigma - 1)/\eta > -1$. A computer verification gives $C(-1, 10^{10}) < 51$ which, together with $\operatorname{Re} z/|z|^2 < M\eta/H^2$, gives

$$\left| \operatorname{Re} W \left(\frac{s_k + \delta_m - 1}{\eta} \right) \right| < \frac{w(0)M}{H^2} \eta + \frac{51}{H^3} \eta^3 < 10^{-12} \eta.$$

Therefore, for all $k \geq 1$,

$$\operatorname{Re} F(s_k - 1) \leq \left| \sum_{0 \leq m \leq M} \kappa_m W \left(\frac{s_k + \delta_m - 1}{\eta} \right) \right| < 10^{-11} \eta.$$

The result follows from summing over k . $\square$

Lemma 2.14. Suppose $\sigma \in [\sigma_0, 1)$, $\eta \in (0, \eta_0]$ and

$$\frac{1 - \sigma_0}{\eta_0} - 10^{-10} \leq \frac{1 - \sigma}{\eta} \leq 1.$$

Let F be as defined in Definition 2.2 and a_0, a_1 be as defined in (2.34). Then

$$\begin{aligned} a_1 F(\sigma - 1 + \eta) + a_1 F(\sigma - \eta) - a_0 F(\sigma - 1) \\ \geq C_1((1 - \sigma)/\eta) + 3.909\eta + 26\eta^2 - 3897\eta^3, \end{aligned}$$

where $C_1(x) := 0.87637 + 0.12002x + 0.01017x^2 - 0.00073x^3$.

Proof. For convenience, denote

$$\mu := \frac{1 - \sigma}{\eta}, \quad \mu_0 := \frac{1 - \sigma_0}{\eta_0} - 10^{-10} = 0.91198\dots \quad (2.38)$$

First we bound the expression $a_1 F(\sigma - 1 + \eta) - a_0 F(\sigma - 1)$, which may be written as

$$\sum_{0 \leq m \leq M} \kappa_m (a_1 W(\delta_m/\eta - \mu + 1) - a_0 W(\delta_m/\eta - \mu)).$$

For any a, ε , with $a \neq 0$ and $a + \varepsilon \neq 0$ one has

$$\frac{1}{a + \varepsilon} = \frac{1}{a} - \frac{\varepsilon}{a^2} + \frac{\varepsilon^2}{a^2(a + \varepsilon)} \quad (2.39)$$

so that

$$W(\delta_m/\eta + x) = \frac{\eta w(0)}{\delta_m + x\eta} + W_0\left(\frac{\delta_m}{\eta} + x\right) = \frac{w(0)}{\delta_m} \eta - \frac{xw(0)}{\delta_m^2} \eta^2 + E(\eta)$$

where

$$E(\eta) = \frac{x^2 w(0)}{\delta_m^2 (\delta_m + x\eta)} \eta^3 + W_0\left(\frac{\delta_m}{\eta} + x\right).$$

If $|x| \leq x_0$ and $\delta_m + x\eta \geq x_1 > 0$, then by Lemma 2.3,

$$|E(\eta)| \leq \left(\frac{x_0^2 w(0)}{(2\sigma_0 - 1)^2 m^2 x_1} + \frac{C(x_1, x_1)}{x_1^3} \right) \eta^3.$$

In particular, if $m \geq 1$ and $x = -\mu$ then we may take $x_0 = \mu_0$ and $x_1 = (2\sigma_0 - 1)m - \mu_0\eta_0$

so that, writing $S_k := \sum_{1 \leq m \leq M} \kappa_m / m^k$,

$$\sum_{1 \leq m \leq M} \kappa_m W(\delta_m/\eta - \mu) \leq \frac{w(0)\eta}{2\sigma - 1} S_1 - \frac{\mu w(0)\eta^2}{(2\sigma - 1)^2} S_2 + 1124\eta^3.$$

Similarly, if $x = -\mu + 1$ then we may take $x_0 = 1 - \mu_0$ and $x_1 = (2\sigma_0 - 1)m - (1 - \mu_0)\eta_0$ to get

$$\sum_{1 \leq m \leq M} \kappa_m W(\delta_m/\eta - \mu + 1) \geq \frac{w(0)\eta}{2\sigma - 1} S_1 - \frac{(\mu + 1)w(0)\eta^2}{(2\sigma - 1)^2} S_2 - 1185\eta^3.$$

We explicitly evaluate $S_1 = -0.689736127\dots$ and $S_2 = -0.818779265\dots$. Furthermore, substituting the values of a_0 , a_1 and $w(0)$, and applying $\mu \geq \mu_0$ and $\sigma \geq \sigma_0$, one finds

$$\begin{aligned} \sum_{1 \leq m \leq M} \kappa_m (a_1 W(\delta_m/\eta - \mu + 1) - a_0 W(\delta_m/\eta - \mu)) \\ > \eta \frac{w(0)(a_1 - a_0)}{2\sigma - 1} S_1 - \eta^2 \frac{w(0)((a_1 - a_0)\mu + a_1)}{(2\sigma - 1)^2} S_2 - 3188\eta^3 \\ > -2.939\eta + 11\eta^2 - 3188\eta^3. \end{aligned} \quad (2.40)$$

Next, the term corresponding to $m = 0$ is

$$a_1 W(-\mu + 1) - a_0 W(-\mu) = \int_0^{2\theta \cot \theta} e^{\mu u} (a_1 e^{-u} - a_0) w(u) du.$$

For all $0 \leq u \leq 2\theta \cot \theta = 1.05923293\dots$ one has

$$\left| 1 + x + \frac{x^2}{2} + \frac{x^3}{6} - e^x \right| < \frac{1}{18} x^4$$

so that

$$a_1 W(-\mu + 1) - a_0 W(-\mu) > c_0 + c_1 \mu + \frac{c_2}{2} \mu^2 + \left(\frac{c_3}{6} - \frac{c^*}{18} \right) \mu^3$$

where

$$c_n := \int_0^{2\theta \cot \theta} u^n (a_1 e^{-u} - a_0) w(u) du, \quad c^* := \int_0^{2\theta \cot \theta} u^3 |a_1 e^{-u} - a_0| w(u) du$$

are computable constants. In particular, we find, using computer assistance,

$$\begin{aligned} c_0 &= 0.8763706262\dots, & c_1 &= 0.1200272738\dots, & c_2 &= 0.0203537951\dots, \\ c_3 &= 0.0004382722\dots, & c^* &= 0.0190417514\dots, \end{aligned}$$

so that

$$a_1 W(-\mu + 1) - a_0 W(-\mu) > C_1(\mu). \quad (2.41)$$

It remains to bound $a_1 F(\sigma - \eta)$. One has

$$F(\sigma - \eta) = \sum_{0 \leq m \leq M} \kappa_m W((\sigma - \eta + \delta_m)/\eta).$$

Applying (2.39) with $a = 1 + \delta_m$ and $\varepsilon = \sigma - 1 - \eta = -(\mu + 1)\eta$,

$$W\left(\frac{\sigma + \delta_m - \eta}{\eta}\right) = \eta w(0) \left(\frac{1}{1 + \delta_m} + \frac{1 + \mu}{(1 + \delta_m)^2} \eta \right) + E_1(\eta)$$

where

$$E_1(\eta) = \frac{w(0)(1 + \mu)^2}{(1 + \delta_m)(1 + \delta_m - (1 + \mu)\eta)} \eta^3 + W_0\left(\frac{\sigma + \delta_m - \eta}{\eta}\right).$$

Recall that $\mu_0 \leq \mu \leq 1$, $\eta \leq \eta_0$ and $\delta_m \geq d_m := (2\sigma_0 - 1)m$. Furthermore, for all $m \geq 0$, one has $(\sigma + \delta_m - \eta)/\eta \geq \sigma_0/\eta_0 - 1$ so that by Lemma 2.3

$$|E_1(\eta)| \leq \left(\frac{4w(0)}{(1 + d_m)(1 + d_m - (1 + \mu_0)\eta_0)m} + \frac{C(\sigma_0/\eta_0 - 1, \sigma_0/\eta_0 - 1)}{(\sigma_0 - \eta_0)^3} \right) \eta^3.$$

Next we write $T_k(\sigma) := \sum_{0 \leq m \leq M} \kappa_m / (1 + \delta_m)^k$, so that

$$T'_k(\sigma) = -2k \sum_{1 \leq m \leq M} \frac{m\kappa_m}{(1 + \delta_m)^{k+1}}$$

is positive for $k = 1, 2$ and all $\sigma_0 \leq \sigma \leq 1$. Therefore,

$$\begin{aligned} a_1 F(\sigma - \eta) &= a_1 \eta w(0) (T_1(\sigma) + \eta(1 + \mu)T_2(\sigma)) + a_1 E_1(\eta) \\ &\geq a_1 \eta w(0) T_1(\sigma_0) + a_1 \eta^2 w(0) (1 + \mu_0) T_2(\sigma_0) + a_1 E_1(\eta) \\ &> 6.848\eta + 15\eta^2 - 709\eta^3. \end{aligned}$$

The result follows from combining this bound with (2.40) and (2.41). $\square$

We now proceed to the main inductive argument.

Lemma 2.15 (Iteration lemma). *Suppose that $\zeta(\sigma + it) \neq 0$ in the region*

$$\sigma > 1 - \frac{A}{\log t}, \quad t \geq H$$

for some $1/6 < A \leq A_0$. Then $\zeta(\sigma + it) \neq 0$ in the region

$$\sigma > 1 - \frac{A + \varepsilon}{\log t}, \quad t \geq H$$

with $\varepsilon = 10^{-100}$.

Proof. Suppose for a contradiction that $\rho_0 = \beta_0 + it$ is a zero of $\zeta(s)$ with

$$A \leq (1 - \beta_0) \log t < A + \varepsilon \quad \text{and} \quad t \geq H. \quad (2.42)$$

Fix $s_k = \sigma + ikt$ with $0 \leq k \leq K$ and $\sigma = 1 - A/\log(Kt + T_0)$, so that $\sigma \in [\sigma_0, 1)$. Let

$$\eta := 1 - \beta_0$$

so that $\eta \in (0, \eta_0]$, and choose

$$f(u) = \eta w(\eta u) \sum_{0 \leq m \leq M} \kappa_m e^{-\delta_m u}$$

as in Definition 2.2. By Lemma 2.8 with $s = s_k$,

$$\begin{aligned} \operatorname{Re} \sum_{n \geq 1} \frac{\Lambda(n)}{n^{s_k}} f(\log n) &= f(0) \sum_{0 \leq m \leq M} \kappa_m \left(-\frac{1}{2} \log \pi + \frac{1}{2} \operatorname{Re} \frac{\Gamma'}{\Gamma} \left(\frac{s_k + \delta_m}{2} + 1 \right) \right) \\ &\quad + \operatorname{Re} F(s_k - 1) - \operatorname{Re} \sum_{\rho} F(s_k - \rho) + E_k, \end{aligned}$$

where $|E_k| \leq 2730\eta^3$ if $k = 0$ and $|E_k| \leq 53\eta^2 + 1605\eta^3$ if $k \geq 1$. Next, let

$$P(x) = \sum_{0 \leq k \leq K} a_k \cos kx$$

be the non-negative trigonometric polynomial constructed in Section 2.4. By Lemma 2.9 and Lemma 2.8 one has

$$\begin{aligned} \frac{f(0)}{12} &\leq \sum_{n \geq 1} \frac{\Lambda(n)}{n^\sigma} f(\log n) P(t \log n) \\ &= \operatorname{Re} \sum_{0 \leq k \leq K} a_k \sum_{n \geq 1} \frac{\Lambda(n)}{n^{s_k}} f(\log n) \\ &= \sum_{0 \leq k \leq K} a_k \left[f(0) \sum_{0 \leq m \leq M} \kappa_m \left(-\frac{1}{2} \log \pi + \frac{1}{2} \operatorname{Re} \frac{\Gamma'}{\Gamma} \left(\frac{s_k + \delta_m}{2} + 1 \right) \right) \right. \\ &\quad \left. + \operatorname{Re} F(s_k - 1) - \operatorname{Re} \sum_{\rho} F(s_k - \rho) + E_k \right]. \end{aligned} \quad (2.43)$$

Now we invoke a series of estimates to bound the terms on the right side. First, Lemma 2.12 is used to bound the main term of order $\log t$:

$$\sum_{0 \leq m \leq M} \kappa_m \sum_{0 \leq k \leq K} a_k \left(-\frac{1}{2} \log \pi + \frac{1}{2} \operatorname{Re} \frac{\Gamma'}{\Gamma} \left(\frac{s_k}{2} + 1 \right) \right) \leq \frac{a\kappa}{2} \log t - 1.568, \quad (2.44)$$

where $\kappa := \sum_{0 \leq m \leq M} \kappa_m = 433/859$. Next, by Lemma 2.13 one has

$$\operatorname{Re} \sum_{0 \leq k \leq K} a_k F(s_k - 1) \leq a_0 F(\sigma - 1) + 10^{-10} \eta \quad (2.45)$$

and by Lemma 2.5 one has

$$\operatorname{Re} \sum_{0 \leq k \leq K} a_k \sum_{\rho} F(s_k - \rho) > F(\sigma - \beta_0) + F(\sigma - 1 + \beta_0) - 10^{-7}. \quad (2.46)$$

Finally, Lemma 2.8 gives a bound for the error term

$$\left| \sum_{0 \leq k \leq K} a_k E_k \right| < 187\eta^2 + 8387\eta^3. \quad (2.47)$$

Substituting (2.45), (2.46), (2.44) and (2.47) into (2.43), we have

$$\begin{aligned} 0 \leq & \frac{a\kappa}{2} f(0) \log t + a_0 F(\sigma - 1) - a_1 (F(\sigma - \beta_0) + F(\sigma - 1 + \beta_0)) \\ & + 10^{-7} + (-w(0)/12 - 1.568w(0) + 10^{-10})\eta + 187\eta^2 + 8387\eta^3. \end{aligned}$$

At this point we note that by our assumption (2.42) (recalling that $\eta = 1 - \beta_0$, $A > 1/6$ and μ_0 is as defined in (2.38)), we have

$$\mu := \frac{1 - \sigma}{\eta} > \frac{A}{A + \varepsilon} \frac{\log t}{\log(Kt + T_0)} > \mu_0$$

and also $\mu < 1$, so that we may apply Lemma 2.14 to obtain

$$\frac{a\kappa}{2} f(0) \log t \geq C_1(\mu) + C_2(\eta) - 10^{-7}$$

where C_1 is defined in Lemma 2.14 and $C_2(\eta) := 13.27\eta - 161\eta^2 - 12284\eta^3$. Note that C_1 and C_2 are positive and increasing on $[\mu_0, 1]$ and $[0, \eta_0]$ respectively. Thus we may use (in light of $A > 1/6$)

$$\mu > \frac{A}{A + \varepsilon} \frac{\log t}{\log(K + T_0/H) + \log t} > 1 - \frac{2.78}{\log t} \quad \text{and} \quad \eta > \frac{1}{6 \log t}$$

to obtain (with the aid of any symbolic algebra package)

$$\begin{aligned} \frac{a\kappa}{2} f(0) \log t & > C_1 \left(1 - \frac{2.78}{\log t} \right) + C_2 \left(\frac{1}{6 \log t} \right) - 10^{-7} \\ & > 1.00583 + \frac{1.8275}{\log t} - \frac{4.4106}{(\log t)^2} - \frac{56.855}{(\log t)^3}. \end{aligned}$$

The right side is decreasing for $t \in [H, H_1]$ so we may lower-bound this expression by its value at $t = H_1$, which is $1.036428\dots$. Therefore

$$\eta \log t > \frac{1.03642}{a\kappa w(0)/2} = 0.205677\dots > A_0 + \varepsilon \geq A + \varepsilon,$$

as required. □

To complete the proof of Lemma 2.1 it remains to note that all zeroes $\beta + i\gamma$ with $3 \leq \gamma < H$ are known to lie on the critical line thanks to the computational verification performed in [PT21], so Lemma 2.1 follows in this region. For $\gamma \geq H$, we use Lemma 2.15 combined with any existing zero-free region of the form $\sigma > 1 - c/\log t$ for some $c > 1/6$, e.g. those proved in [Kad05], [MT14] or [MTY24].

Chapter 3

Littlewood's zero-free region

3.1 Introduction

In 1922, Littlewood [Lit22] showed that for sufficiently large t , $\zeta(\sigma + it) \neq 0$ in the region

$$\sigma > 1 - A \frac{\log \log t}{\log t} \tag{3.1}$$

for some absolute constant $A > 0$, thereby enlarging the “classical” zero-free region by a factor of $\log \log t$. The key input that facilitates this improvement is a non-trivial upper bound on $|\zeta(s)|$ inside the critical strip, which is obtained from certain exponential sum estimates. A number of tools may then be used to pass from estimates of $\zeta(s)$ to a zero-free region, most notably the “local” method of Landau.

The goal of this chapter is to prove an explicit version of this result. Specifically, we will prove Theorem 1.2, which follows from combining Lemma 2.1 from the previous chapter with the following lemma:

Lemma 3.1. *If $t \geq \exp(56.5)$ then $\zeta(\sigma + it) \neq 0$ for*

$$\sigma > 1 - \frac{1}{19.62} \frac{\log \log t}{\log t}.$$

The arguments in this chapter are a refinement of those in an earlier manuscript [Yan24] by the author, where a weaker zero-free region was obtained with a constant of $(21.233)^{-1}$. The proof requires two main ingredients. The first ingredient is the following explicit estimate of $\zeta(s)$ along certain vertical lines inside the critical strip.

Lemma 3.2 (Yang [Yan24] Theorem 1.1). *For an integer $k \geq 4$ and $\sigma_k := 1 - k/(2^k - 2)$, one has*

$$|\zeta(\sigma_k + it)| \leq 1.546t^{1/(2^k-2)} \log t \quad (t \geq 3).$$

The second ingredient is a method of detecting zeroes due to Ford [For02b] (Lemma 3.4 below), which replaces Landau's method in our treatment. Landau's approach involves taking an integral on a small circle centered at s , while Ford takes the integral on two vertical lines, which makes more efficient use of our knowledge of $\zeta(s)$.

The analysis in [Yan24] focuses on deriving the first ingredient; Ford's method was then applied with minimal changes. In contrast, in this chapter we shall leave Lemma 3.2 unchanged, and focus on refinements to the zero-detection method. In particular, a number of ideas we encountered in the previous chapter can be directly imported into Ford's argument, such as evaluating the zero-detector at a point s inside the critical strip (instead of on the line $\operatorname{Re} s = 1$) and using an iterative argument to progressively enlarge the zero-free region. Furthermore we take advantage of more precise tail estimates of the form $F_0(z) \ll |z|^{-3}$ (compared to $F_0(z) \ll |z|^{-2}$ used in [For02b]) which provide sharper estimates of various error terms. Our argument unifies some of the ideas and approaches presented in [Kad05] and [For02b].

3.2 The zero detector

We begin by recalling Ford's zero-detector, which is the main tool we use to relate upper bounds on $\zeta(s)$ to the behaviour of zeroes of $\zeta(s)$ close to the line $\operatorname{Re} s = 1$.

Lemma 3.3 (Ford [For02b] Lemma 2.2). *Let f be the quotient of two entire functions of finite order. Suppose f has no pole or zero at $z = s$ and $z = 0$. For all $\lambda > 0$, except for*

a set of Lebesgue measure 0 (possibly depending on f and s), we have

$$\begin{aligned} -\operatorname{Re} \frac{f'(s)}{f(s)} &= \operatorname{Re} \frac{\pi}{2\lambda} \sum_{|\operatorname{Re}(s-\rho)| \leq \lambda} m_\rho \cot \left(\frac{\pi(\rho-s)}{2\lambda} \right) \\ &\quad + \frac{1}{4\lambda} \int_{-\infty}^{\infty} \frac{\log |f(s-\lambda+2\lambda iu/\pi)| - \log |f(s+\lambda+2\lambda iu/\pi)|}{(\cosh u)^2} du \end{aligned}$$

where ρ runs through all zeroes and poles of f (with multiplicity) and $m_\rho = 1$ if ρ is a zero of f , or -1 if ρ is a pole of f .

One could in principle use the above lemma directly, with $f(s) = \zeta(s)$, along with the arguments in the rest of this chapter, to construct a zero-free region of the form (3.1). However, to obtain better constants we instead work with $\zeta'(s)/\zeta(s)$ “mollified” by a smoothing function. This also ultimately allows us to take s inside the critical strip. The choice of smoothing function is similar to that in Definition 2.2, except that one no longer has access to Stechkin’s device because in Lemma 3.3 we are only summing over a subset of all complex zeroes of $\zeta(s)$, so that the sum no longer contains both the zeroes ρ and $1-\bar{\rho}$. Nevertheless, one may still readily apply Heath-Brown’s [HB92a] family of smoothing functions. We start by recalling the following function $w(u)$ from Definition 2.1 of the previous chapter, reproduced here for convenience:

Definition 3.1 (The functions w , W and W_0). *For fixed $\theta \in (0, \pi/2)$, let w be as defined as*

$$\begin{aligned} w(u) &:= \sec^2 \theta (\sec^2 \theta (\theta \cot \theta - u/2) \cos(u \tan \theta) + 2\theta \cot \theta - u \\ &\quad + \sin(2\theta - u \tan \theta) \csc 2\theta - 2(1 + \sin(\theta - u \tan \theta) \csc \theta)) \end{aligned}$$

for $0 \leq \theta \leq 2\theta \cot \theta$, and $w(u) := 0$ elsewhere. Let $W(z) := \int_0^\infty e^{-zu} w(u) du$ denote its Laplace transform, and let $W_0(z) := W(z) - w(0)/z$.

Definition 3.2 (The functions f , F and F_0). *Let $w(u)$ be defined in Definition 3.1 with respect to some fixed $0 < \theta < \pi/2$. For fixed $\eta > 0$, we define the functions $f : \mathbb{R} \rightarrow \mathbb{R}$, $F : \mathbb{C} \rightarrow \mathbb{C}$ and $F_0 : \mathbb{C} \rightarrow \mathbb{C}$ as*

$$f(u) := \eta w(\eta u), \quad F(z) := \int_0^\infty e^{-zu} f(u) du, \quad F_0(z) := F(z) - \frac{f(0)}{z}.$$

By inspection, one has $F(z) = W(z/\eta)$ and $F_0(z) = W_0(z/\eta)$. Furthermore, we recall that one has the estimate

$$|W_0(z)| \leq C(\nu, r)|z|^{-3} \quad (\operatorname{Re} z \geq \nu, |z| \geq r), \quad (3.2)$$

where $C(\nu, r)$ is defined in Lemma 2.3.

Lemma 3.4 (Zero-detector for $\zeta(s)$). *Let $s = \sigma + it$ where $3/4 < \sigma < 1$ and $\zeta(s) \neq 0$. For all $1 - \sigma < \lambda < 1/2$ except a set of Lebesgue measure zero, one has*

$$\operatorname{Re} \sum_{n \geq 1} \frac{\Lambda(n)}{n^s} f(\log n) = I(s, \lambda)$$

where

$$\begin{aligned} I(s, \lambda) &:= \frac{f(0)}{4\lambda} \int_{-\infty}^{\infty} \frac{\log |\zeta(s - \lambda + 2\lambda iu/\pi)| - \log |\zeta(s + \lambda + 2\lambda iu/\pi)|}{(\cosh u)^2} du \\ &\quad + \operatorname{Re} F^*(s - 1) - \sum_{\substack{\rho = \beta + i\gamma \\ \beta \geq \sigma - \lambda}} \operatorname{Re} F^*(s - \rho) - \sum_{\substack{\rho = \beta + i\gamma \\ \beta < \sigma - \lambda}} \operatorname{Re} F_0(s - \rho) + E(s), \\ F^*(z) &:= F(z) + f(0) \left(\frac{\pi}{2\lambda} \cot \left(\frac{\pi z}{2\lambda} \right) - \frac{1}{z} \right) \end{aligned}$$

and

$$E(s) := \operatorname{Re} \frac{1}{2\pi i} \int_{1/2 - i\infty}^{1/2 + i\infty} F_0(s - z) \operatorname{Re} \frac{\Gamma'}{\Gamma} \left(\frac{z}{2} \right) dz + \operatorname{Re} F_0(s).$$

Proof. We apply the version of Weil's [Wei52] explicit formula found in [Kad05, Theorem 3.1] with $q = 1$ and

$$\phi(y) = \begin{cases} (f(0) - f(y))e^{-ys}, & y \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

For all complex numbers s not coinciding with a zero or pole of ζ ,

$$\begin{aligned} \operatorname{Re} \sum_{n \geq 1} \frac{\Lambda(n)}{n^s} f(\log n) &= f(0) \left(-\operatorname{Re} \frac{\zeta'(s)}{\zeta(s)} - \operatorname{Re} \frac{1}{s-1} + \sum_{\rho} \operatorname{Re} \frac{1}{s-\rho} \right) \\ &\quad + \operatorname{Re} F(s-1) - \sum_{\rho} \operatorname{Re} F(s-\rho) + \operatorname{Re} E(s), \end{aligned} \quad (3.3)$$

where ρ runs through non-trivial zeroes of ζ . First, we check that all sums over zeroes in (3.3) are absolutely convergent. One has

$$\left| \operatorname{Re} \frac{1}{s-\rho} \right| = \frac{|\sigma - \beta|}{(\sigma - \beta)^2 + (t - \gamma)^2}$$

and $\sum_{\rho}(t - \gamma)^{-2}$ converges, and hence so does $\sum_{\rho}|\operatorname{Re}(s - \rho)^{-1}|$. Furthermore, F is holomorphic with $|\operatorname{Re}F(z)| \ll |z|^{-2}$ uniformly for $|\operatorname{Re} z| \leq 1$ and $|\operatorname{Im} z| \geq 1$, and there are finitely many zeroes ρ for which $|\operatorname{Im}(s - \rho)| < 1$, so by the same reasoning $\sum_{\rho}|\operatorname{Re}F(s - \rho)|$ converges.

We may thus freely change the order of summation over zeroes. Applying $F(z) = f(0)/z + F_0(z)$, we get

$$\operatorname{Re} \sum_{n \geq 1} \frac{\Lambda(n)}{n^s} f(\log n) = -f(0) \operatorname{Re} \frac{\zeta'(s)}{\zeta(s)} + \operatorname{Re} F_0(s - 1) - \sum_{\rho} \operatorname{Re} F_0(s - \rho) + E(s)$$

for all s not coinciding with a zero or pole of zeta. We apply Lemma 3.3 with λ , s and $f = \zeta$. Then for all $\lambda \in (1 - \sigma, 1/2)$, except for a set of Lebesgue measure zero, we have

$$\begin{aligned} -\operatorname{Re} \frac{\zeta'(s)}{\zeta(s)} &= -\operatorname{Re} \frac{\pi}{2\lambda} \sum_{\substack{\rho=\beta+i\gamma \\ \beta \geq \sigma-\lambda}} \cot \left(\frac{\pi(s - \rho)}{2\lambda} \right) + \operatorname{Re} \frac{\pi}{2\lambda} \cot \left(\frac{\pi(s - 1)}{2\lambda} \right) \\ &\quad + \frac{1}{4\lambda} \int_{-\infty}^{\infty} \frac{\log |\zeta(s - \lambda + 2\lambda iu/\pi)| - \log |\zeta(s + \lambda + 2\lambda iu/\pi)|}{(\cosh u)^2} du. \end{aligned} \quad (3.4)$$

Lastly, since $F^*(z)$ is holomorphic for $|z| \leq \lambda$, one has by (3.3) and (3.4)

$$\begin{aligned} \operatorname{Re} \sum_{n \geq 1} \frac{\Lambda(n)}{n^s} f(\log n) &= \frac{f(0)}{4\lambda} \int_{-\infty}^{\infty} \frac{\log |\zeta(s - \lambda + 2\lambda iu/\pi)| - \log |\zeta(s + \lambda + 2\lambda iu/\pi)|}{(\cosh u)^2} du \\ &\quad + \operatorname{Re} F^*(s - 1) - \sum_{\substack{\rho=\beta+i\gamma \\ \beta \geq \sigma-\lambda}} \operatorname{Re} F^*(s - \rho) - \sum_{\substack{\rho=\beta+i\gamma \\ \beta < \sigma-\lambda}} \operatorname{Re} F_0(s - \rho) + E(s) \end{aligned}$$

for all s as in the lemma and for all $\lambda \in (1 - \sigma, 1/2)$ outside a set of Lebesgue measure zero. $\square$

3.3 Proof of Lemma 3.1

Other than the use of a different zero-detector, the proof of Lemma 3.1 is conceptually similar to the approach taken in the previous chapter. Throughout, let

$$H_1 := \exp(56.5), \quad H_2 := \exp(225.4). \quad (3.5)$$

As before, we use an inductive argument. For each

$$0 < A \leq A_1 := (19.62)^{-1} = 0.05096 \dots,$$

let $\mathcal{H}(A)$ denote the hypothesis

$$\mathcal{H}(A) : \quad \zeta(\sigma + it) \neq 0 \quad \text{for} \quad \sigma \geq 1 - A \frac{\log \log t}{\log t}, \quad t \geq H_1.$$

In particular, $\mathcal{H}(A_1)$ implies Lemma 3.1. By the Vinogradov–Korobov zero-free region, $\mathcal{H}(A)$ holds for some $A > 0$. It therefore suffices to show

Proposition 3.1. *For any $0 < A \leq A_1$ and $\varepsilon = 10^{-100}$, one has*

$$\mathcal{H}(A) \implies \mathcal{H}(A + \varepsilon).$$

The proof of Proposition 3.1 is the object of the rest of this chapter. Suppose for sake of contradiction that for some $A \in (0, A_1]$, $\mathcal{H}(A)$ holds but $\mathcal{H}(A + \varepsilon)$ does not. This is equivalent to the following assumption.

Assumption 3.1. *For some $A \in (0, A_1]$, the hypothesis $\mathcal{H}(A)$ holds, and furthermore there exists a zero $\rho_0 = \beta_0 + it$ satisfying*

$$1 - (A + \varepsilon) \frac{\log \log t}{\log t} \leq \beta_0 < 1 - A \frac{\log \log t}{\log t} \quad \text{and} \quad t \geq H_1.$$

We will show that Assumption 3.1 leads to a contradiction. As before, we take

$$\eta := 1 - \beta_0 \tag{3.6}$$

in Definition 3.2. Meanwhile, let $P(x)$ be the non-negative trigonometric polynomial constructed in Section 2.4.¹ For any $x \in \mathbb{R}$, one has

$$P(x) = \sum_{0 \leq k \leq K} a_k \cos(kx) \geq 0 \tag{3.7}$$

¹One could equally use any admissible non-negative trigonometric polynomial here, such as the degree 40 polynomial recorded in [MTY24, Table 2]. As t increases, the cost associated with using a polynomial of higher degree becomes less significant compared to the savings they generate. Nevertheless, for values of t most relevant to this chapter, polynomial (3.7) remains the most favourable (by a small margin). By contrast, in the next chapter we will switch to a polynomial of larger degree for the proof of the Vinogradov–Korobov zero-free region, where t is very large.

where $K = 16$. This property guarantees that for any $t \in \mathbb{R}$

$$0 \leq \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\sigma}} f(\log n) P(t \log n) = \operatorname{Re} \sum_{k=0}^K a_k \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{s_k}} f(\log n).$$

We apply Lemma 3.4 to all terms on the right side with $k \geq 1$, and Lemma 2.6 from the previous chapter to the term with $k = 0$. One obtains, for all $\lambda \in (1 - \sigma, 1/2)$ except a set of Lebesgue measure zero,

$$0 \leq \sum_{1 \leq k \leq K} a_k I(s_k, \lambda) + a_0 Q(\sigma), \quad (3.8)$$

where

$$Q(\sigma) = f(0) \left(-\frac{1}{2} \log \pi + \frac{1}{2} \operatorname{Re} \frac{\Gamma'}{\Gamma} \left(\frac{\sigma}{2} + 1 \right) \right) + \operatorname{Re} F(\sigma - 1) - \operatorname{Re} \sum_{\rho} F(\sigma - \rho) + E(\sigma) \quad (3.9)$$

and $E(\sigma)$ is defined in Lemma 3.4. As before we now seek to upper-bound the right side of (3.8). This requires a more careful treatment than that of the previous chapter, because the ratio of the error and “main” terms decay to zero rather slowly (at a rate of $(\log \log t)^{-1}$) as $t \rightarrow \infty$. To assist in the tracking of constants, we begin by fixing the choices of θ , σ and λ . These choices were made with the benefit of hindsight; however, for clearer presentation we record them before specifying the arguments that justify their choice. Hereafter we take

$$\theta := 1.1272 \quad (3.10)$$

in the definition of $w(u)$ and $f(u)$, which appears close to optimal. In [HB92a], it was shown that the asymptotically optimal choice of θ (as $t \rightarrow \infty$) is given by the unique solution to the equation

$$\sin^2 \theta = \frac{a_1}{a_0} (1 - \theta \cot \theta) \quad (0 < \theta < \pi/2),$$

which for our trigonometric polynomial is approximately 1.136. The difference between these choices of θ is due to the presence of a secondary error term that remains significant for the range of t relevant to our application. This error term arises from the contribution to $I(s_k, \lambda)$ of those hypothetical zeroes close to ρ_0 . For large t , one can use a “local” zero-density estimate to bound the number of such zeroes; however, such bounds are not

sufficiently strong when t is small. Consequently, we divide our argument into two regimes. For $t < H_2$ we estimate the contribution of such zeroes “trivially” using explicit bounds on the zero-counting function $N(T)$. For $t \geq H_2$ we switch over to an argument that makes use of our knowledge of the local density of zeroes.

This two-part argument also influences our choice of σ and λ . Specifically, we eventually choose $\sigma = \sigma^*$ and $\lambda = \lambda^*$, given by:

Definition 3.3 (The parameters σ^* and λ^*). *With K , H_1 and H_2 defined in (3.7) and (3.5), define*

$$\sigma^* := \sigma^*(t) = 1 - A \frac{\log \log(Kt + T_0)}{\log(Kt + T_0)}, \quad T_0 := 10^{10}.$$

Furthermore, let $\lambda^* = \lambda^*(t)$ be defined as

$$\begin{aligned} \lambda^* &:= \left(8 - \frac{24.2}{\log \log t}\right)^{-1} && (H_1 \leq t < \exp(216)), \\ \lambda^* &:= \left(16 - \frac{67.19}{\log \log t}\right)^{-1} && (\exp(216) \leq t < H_2), \\ \lambda^* &:= 2.248 \frac{(\log \log t)^2}{\log t} && (t \geq H_2). \end{aligned}$$

One may verify that the assumptions $\sigma \in (3/4, 1)$ and $\lambda \in (1 - \sigma, 1/2)$ are satisfied by our choices for $t \geq H_1$. The parameter σ^* is chosen so that Assumption 3.1 implies $\zeta(s) \neq 0$ for all $\operatorname{Re} s \in [\sigma^*, 1]$ and $\operatorname{Im} s \in [t, Kt]$. The shape of λ^* is different for $t < H_2$ and $t \geq H_2$, reflecting the two different approaches we take to bound the sum over zeroes appearing in $I(s, \lambda)$.

For convenience, we also define the constants

$$\begin{aligned} \sigma_1 &:= \sigma^*(H_1) = 0.99648\dots, \\ \eta_1 &:= (A_1 + \varepsilon) \log \log H_1 / \log H_1 = 0.00363\dots, \\ \lambda_1 &:= \lambda^*(H_1) = 0.49966\dots, \\ \lambda_2 &:= \lambda^*(H_2) = 0.29275\dots \end{aligned} \tag{3.11}$$

These constants are defined so that

$$\begin{aligned} \lambda^* &\in (0, \lambda_1] & (t \geq H_1), & \lambda^* &\in (0, \lambda_2] & (t \geq H_2), \\ \sigma^* &\in [\sigma_1, 1) & (t \geq H_1), & \eta &\in (0, \eta_1] & (t \geq H_1). \end{aligned}$$

3.3.1 Bounding the main terms

In this section we use Lemma 3.2 to bound the “main” integral term appearing in $I(s, \lambda)$:

$$\frac{1}{2} \int_{-\infty}^{\infty} \frac{\log |\zeta(s - \lambda + 2\lambda i u / \pi)| - \log |\zeta(s + \lambda + 2\lambda i u / \pi)|}{(\cosh u)^2} du.$$

Due to the rapid decay of $(\cosh u)^{-2}$ as $|u| \rightarrow \infty$, this integral is morally $\log |\zeta(s - \lambda)| - \log |\zeta(s + \lambda)|$. Since $\operatorname{Re}(s - \lambda) < 1 < \operatorname{Re}(s + \lambda)$ by assumption, we can estimate this integral using two pointwise estimates of $\zeta(s)$, one (slightly) inside the critical strip and one outside the critical strip. For the first estimate we use Theorem 3.2, and for the second bound we use an estimate due to Ramaré [Ram15, Lemma 5.4] that

$$|\zeta(\sigma + it)| \leq \frac{e^{\gamma(\sigma-1)}}{\sigma-1} \quad (\sigma > 1) \quad (3.12)$$

where $\gamma = 0.57721 \dots$ is the Euler–Mascheroni constant.

With our choices of σ and λ , this integral generates a leading term of size $O(\log t / \log \log t)$. In comparison, the main term in the previous chapter comes from $\operatorname{Re} \Gamma' / \Gamma(s/2 + 1) \sim \log t$, and this saving of a factor of $\log \log t$ explains the corresponding enlargement of the zero-free region by the same factor.

For optimization purposes it is convenient to have an upper bound on $\zeta(s)$ that holds uniformly inside the critical strip instead of just along a discrete set of vertical lines. To do this we interpolate the bounds on $\zeta(s)$ along two neighbouring vertical lines using the Phragmén–Lindelöf principle (Lemma 3.5 below).

Lemma 3.5 (Fiori [Fio25] Corollary 3). *Let $s = \sigma + it$ and suppose that for $\sigma_1 \leq \sigma \leq \sigma_2$, $f(s)$ is holomorphic and satisfies $|f(s)| \ll \exp(c \exp(c_1 |t|))$ for some $c > 0$ and $0 < c_1 < \pi/(\sigma_2 - \sigma_1)$. Furthermore suppose that for $s_\ell = \sigma_\ell + it$ ($\ell = 1, 2$) we have*

$$|f(s_\ell)| \leq A_\ell |Q_1 + s_\ell|^{\alpha_\ell} |\log(Q_2 + s_\ell)|^\beta |\log \log(Q_3 + s_\ell)|^\gamma$$

where A_ℓ, α_ℓ ($\ell = 1, 2$), Q_j ($1 \leq j \leq 3$) and β, γ jointly satisfies $\alpha_1 \geq \alpha_2$, $Q_1 + \sigma_1 > 0$, $Q_2 + \sigma_1 > 1$ and $Q_3 + \sigma_1 > e$. Then, for all $\operatorname{Re} s \in [\sigma_1, \sigma_2]$,

$$|f(s)| \leq A_1^{w_1} A_2^{w_2} |Q_1 + s|^{\alpha_1 w_1 + \alpha_2 w_2} |\log(Q_2 + s)|^\beta |\log \log(Q_3 + s)|^\gamma$$

where $w_1 := (\sigma_2 - \sigma)/(\sigma_2 - \sigma_1)$ and $w_2 := (\sigma - \sigma_1)/(\sigma_2 - \sigma_1)$.

We use the above lemma to interpolate between two bounds from Lemma 3.2.

Lemma 3.6. For integers $n \geq 3$, let $\sigma_n = 1 - n/(2^n - 2)$. If $\sigma_n \leq \sigma \leq \sigma_{n+1}$ and $t \geq H_1/2$, then

$$|\zeta(\sigma + it)| \leq C_n t^\mu \log t, \quad \mu = \frac{(1 - \sigma)2^n - 1}{(n - 1)2^n + 2}.$$

where $C_3 = 0.0728 \cdot 2.502^{14\sigma/3}$ and $C_n = 1.549$ for $n \geq 4$.

Proof. Let $z_n := \sigma_n + it$, $\mu_n = 1/(2^n - 2)$ and suppose for $j \in \{n, n + 1\}$,

$$|\zeta(z_j)| \leq c_j t^{\mu_j} \log t \quad (t \geq 3) \tag{3.13}$$

for some constants $c_j > 1/2$. Note that $1/2 \leq \sigma_n < 1$. First, we verify computationally that on $\{s \in \mathbb{C} : 1/2 \leq \operatorname{Re} s \leq 1, |\operatorname{Im} s| \leq 3\}$, one has $|(s - 1)\zeta(s)| < Q$ and $|\log(Q + s)| > 1$ with $Q = 3$. Furthermore, since $c_j > 1/2$, this implies

$$|(z_j - 1)\zeta(z_j)| < c_j |Q + z_j|^{1 + \mu_j} |\log(Q + z_j)| \quad (|t| \leq 3).$$

Meanwhile by (3.13), and since $|\log(Q + z_j)| \geq \log |Q + z_j| > \log t$ and $|z_j - 1|, t < |Q + z_j|$, one in fact has

$$|(z_j - 1)\zeta(z_j)| \leq c_j |Q + z_j|^{1 + \mu_j} |\log(Q + z_j)| \quad (t \in \mathbb{R}).$$

Taking $j = n, n + 1$ and applying Lemma 3.5 gives

$$|\zeta(s)| \leq \frac{c_n^{w_n} c_{n+1}^{w_{n+1}} |Q + s|^{w_n \mu_n + w_{n+1} \mu_{n+1}} |\log(Q + s)|}{|s - 1|} \tag{3.14}$$

for all $\sigma_n \leq \sigma \leq \sigma_{n+1}$, where $w_n := (\sigma_{n+1} - \sigma)/(\sigma_{n+1} - \sigma_n)$ and $w_{n+1} = 1 - w_n$. Next, for $z = x + iy$ with $x > 0, y > 1$, we have the crude bounds

$$|z| < x + y \leq (1 + \varepsilon_0)y,$$

$$\begin{aligned} |\log z| &= |\log |z| + i \arctan(y/x)| < ((\log y + \log(1 + \varepsilon_0))^2 + \pi^2)^{1/2} \\ &< ((1 + \varepsilon_1)^2 + \varepsilon_2)^{1/2} \log y, \end{aligned}$$

where $\varepsilon_0 = x/y$, $\varepsilon_1 = \log(1 + \varepsilon_0)/\log y$ and $\varepsilon_2 = (\pi/\log y)^2$. Taking $z = Q + s$ and using $t \geq H_1/2$ gives $\varepsilon_0, \varepsilon_1 < 10^{-10}$ and $\varepsilon_2 < 2.5 \cdot 10^{-3}$. Applying these inequalities to (3.14), along with $|s - 1| \geq t$ and $\mu_n \leq 1/6$, gives

$$|\zeta(s)| \leq ct^{w_n \mu_n + w_{n+1} \mu_{n+1}} \log t \quad (t \geq H_1/2)$$

where

$$c = (1 + \varepsilon_0)^{1/6} ((1 + \varepsilon_1)^2 + \varepsilon_2)^{1/2} c_n^{w_n} c_{n+1}^{w_{n+1}}.$$

If $n \geq 4$, by Lemma 3.2 we may take $c_n = c_{n+1} = 1.546$ so that $c \leq 1.549$. If $n = 3$, then we use the following estimate due to [HPY24]:

$$|\zeta(1/2 + it)| \leq 0.618 t^{1/6} \log t \quad (t \geq 3)$$

so we may take $c_3 = 0.618$. Combined with $c_4 = 1.546$ from Lemma 3.2, one has

$$c \leq 0.0728 \cdot 2.502^{14\sigma/3}.$$

This completes the proof. □

The next lemma makes explicit the notion that the integral term can be bounded effectively using a pointwise estimate of $\zeta(s)$ inside the critical strip. This lemma is largely the same as [For02b, Lemma 3.4], the main difference being that we only require a bound on $\zeta(\sigma + it)$ to hold for large t instead of for all $t \geq 3$. This technical change allows us to avoid applying the Phragmén–Lindelöf principle at very small values of t where it is poorly suited.

Lemma 3.7. *Let $0 < \delta \leq 1/2$, $t \geq 1000$ and $1/2 \leq \sigma \leq 1 - t^{-1}$ be fixed, and suppose that*

$$|\zeta(\sigma + iy)| \leq X y^Y (\log y)^Z \quad (y \geq t/2)$$

for some constants $X, Y > 0$ and $Z \geq 1/2$. Then

$$\frac{1}{2} \int_{-\infty}^{\infty} \frac{\log |\zeta(\sigma + it + iu\delta)|}{(\cosh u)^2} du \leq \log X + Y \log t + Z \log \log t.$$

Proof. We follow essentially the same argument as [For02b, Lemma 3.4] by splitting the integral into three parts. For convenience let us write

$$q := t/\delta$$

so that $q \geq 2000$. Let

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\log |\zeta(\sigma + it + iu\delta)|}{(\cosh u)^2} du &= \int_{-\infty}^{-3q/2} + \int_{-3q/2}^{-q/2} + \int_{-q/2}^{\infty} \\ &= I_1 + I_2 + I_3, \end{aligned}$$

say. If $u \leq -3q/2$ then $-(t + u\delta) \geq t/2$. By the lemma's assumption and applying $|\zeta(s)| = |\overline{\zeta(\overline{s})}|$ and $\log(t + x) \leq \log t + x/t$,

$$\begin{aligned} \log |\zeta(\sigma + i(t + u\delta))| &\leq \log X + Y \log(-t - u\delta) + Z \log \log(-t - u\delta) \\ &\leq \log(Xt^Y(\log t)^Z) - \left(Y + \frac{Z}{\log t}\right) \left(\frac{u}{q} + 2\right). \end{aligned}$$

Furthermore, using $(\cosh u)^2 > e^u/4$ (for $u > 0$) and $\delta \leq 1/2$,

$$- \int_{-\infty}^{-3q/2} \frac{u/q + 2}{(\cosh u)^2} du < 4 \int_q^{\infty} (u/q + 2)e^{-2u} du = (6 + 1/q)e^{-2q}.$$

Therefore,

$$I_1 \leq \log(Xt^Y(\log t)^Z) \int_{-\infty}^{-3q/2} \frac{du}{(\cosh u)^2} + (6 + 1/q)e^{-2q}. \quad (3.15)$$

If $-3q/2 \leq u \leq -q/2$ then $|t + u\delta| \leq t/2$. Writing $s = \sigma + i(t + u\delta)$, we have $|s - 1| \geq |\operatorname{Re}(s - 1)| \geq t^{-1}$ by assumption. Also, $|s| < t/2 + 1$. Since $\sigma \geq 1/2$ and $t \geq 1000$ we have, via a classical identity (see e.g. Titchmarsh [Tit86, (2.1.4)]),

$$|\zeta(s)| = \left| \frac{1}{s-1} + \frac{1}{2} + s \int_1^{\infty} \frac{[x] - x + 1/2}{x^{s+1}} dx \right| \leq \frac{1}{|s-1|} + \frac{1}{2} + \frac{|s|}{2} \int_1^{\infty} \frac{dx}{x^{\sigma+1}} < 2t,$$

so that, since $t = \delta q \leq q/2$ and $(\cosh u)^2 > \exp(2|u|)/4$,

$$I_2 \leq \log(2t) \int_{-3q/2}^{-q/2} \frac{du}{(\cosh u)^2} < 4e^{-q}q \log q < e^{-q/2}. \quad (3.16)$$

Finally, if $u \geq -q/2$, then $t + u\delta \geq t/2$. By the lemma's assumption, and applying $\log(1 + x) \leq x$ and $\log(1 + x) \leq x - \frac{1}{2}x^2 + \frac{1}{3}x^3$ gives

$$\begin{aligned} \log |\zeta(\sigma + i(t + u\delta))| &\leq \log X + Y \log(t + u\delta) + Z \log \log(t + u\delta) \\ &\leq \log(Xt^Y(\log t)^Z) + \left(Y + \frac{Z}{\log t}\right) \left(\frac{u\delta}{t} - \frac{(u\delta)^2}{2t^2} + \frac{(u\delta)^3}{3t^3}\right). \end{aligned}$$

Applying $x + \frac{1}{2}x^2 + \frac{1}{3}x^3 < 10x^3$ (for $x > \frac{1}{2}$), $(\cosh u)^2 > \frac{1}{4}e^{2u}$ (for $u > 0$) and $q \geq 2000$ yields

$$\begin{aligned} \int_{-q/2}^{\infty} \frac{u/q - (u/q)^2/2 + (u/q)^3/3}{(\cosh u)^2} du &= \int_{-\infty}^{\infty} - \int_{-\infty}^{-q/2} \\ &= -\frac{\pi^2}{12q^2} + \int_{q/2}^{\infty} \frac{u/q + (u/q)^2/2 + (u/q)^3/3}{(\cosh u)^2} du \\ &\leq -\frac{\pi^2}{12q^2} + \frac{40}{q^3} \int_{1000}^{\infty} u^3 e^{-2u} du < -\frac{\pi^2}{12q^2} + \frac{10^{-100}}{q^3}, \end{aligned}$$

so that

$$I_3 \leq \log(Xt^Y(\log t)^Z) \int_{-q/2}^{\infty} \frac{du}{(\cosh u)^2} - \frac{\pi^2}{12q^2} + \frac{10^{-100}}{q^3}. \quad (3.17)$$

Combining (3.15), (3.16) and (3.17), and since $\int_{-\infty}^{\infty} (\cosh u)^{-2} du = 2$, we have

$$\int_{-\infty}^{\infty} \frac{\log |\zeta(\sigma + i(t + u\delta))|}{(\cosh u)^2} du < 2 \log(Xt^Y(\log t)^Z) + R,$$

where (recalling that $q \geq 2000$ and $t = q\delta \leq q/2$)

$$\begin{aligned} R &= e^{-q/2} + \left(Y + \frac{Z}{\log t}\right) \left((6 + 1/q)e^{-2q} - \frac{\pi^2}{12q^2} + \frac{10^{-100}}{q^3}\right) \\ &< e^{-q/2} - \frac{1}{4q^2 \log q} < 0, \end{aligned}$$

as required. $\square$

The next two lemmas are the main results of this section.

Lemma 3.8 (Main term estimate for small t). *Let $H_1 \leq t \leq H_2$ and suppose σ^* and λ^* are as in Definition 3.3. If $s_k^* = \sigma^* + ikt$ and $0 < \delta \leq 1/2$ is a constant, then*

$$\begin{aligned} \frac{1}{4\lambda} \sum_{1 \leq k \leq K} a_k \int_{-\infty}^{\infty} \frac{\log |\zeta(s_k^* - \lambda^* + iu\delta)| - \log |\zeta(s_k^* + \lambda^* + iu\delta)|}{(\cosh u)^2} du \\ \leq 3.4431 \frac{\log t}{\log \log t}. \end{aligned}$$

Proof. Throughout, let $T = kt$ for integers $1 \leq k \leq K$ and suppose $\sigma_n := 1 - n/(2^n - 2)$ for integers $n \geq 3$. For convenience, let us denote $(T_3, T_4, T_5) = (H_1, e^{216}, H_2)$ and $(E_3, E_4) = (24.2, 67.19)$, so that for $n = 3, 4$ one has

$$\lambda^*(t) = \left(2^n - \frac{E_n}{\log \log t}\right)^{-1} \quad (T_n \leq t \leq T_{n+1}).$$

The constants E_n are chosen so that $\sigma^*(t) + \lambda^*(t) \in [\sigma_n, \sigma_{n+1}]$ for all $t \in [T_n, T_{n+1}]$.

Writing $\delta_n := 1/((n-1)2^n + 2)$ and applying Lemma 3.6 and Lemma 3.7,

$$\begin{aligned} & \frac{1}{4\lambda} \int_{-\infty}^{\infty} \frac{\log |\zeta(s_k^* - \lambda^* + iu\delta)|}{(\cosh u)^2} du \\ & \leq \frac{1}{2\lambda^*} (\delta_n(2^n(1 - \sigma^* + \lambda^*) - 1) \log T + \log \log T + \log C_n) \\ & = 2^{n-1} \delta_n \frac{1 - \sigma^*}{\lambda^*} \log T + \frac{\delta_n E_n}{2} \frac{\log T}{\log \log t} + \frac{\log \log T + \log C_n}{2\lambda^*}. \end{aligned}$$

However, for $n = 3, 4$ and $T_n \leq t \leq T_{n+1}$ one may verify that

$$\frac{2^{n-1} \delta_n (\log \log t)^2}{\lambda^* \log t} = \frac{2^{n-1} \delta_n \log \log t (2^n \log \log t - E_n)}{\log t} < 0.133$$

so that, since $\log T < \log(Kt + T_0)$ and $\log \log(Kt + T_0) < 1.012 \log \log t$ for $t \geq H_1$, and $A \leq A_1$ by assumption,

$$2^{n-1} \delta_n \frac{1 - \sigma^*}{\lambda^*} \log T < 0.133(A_1 + \varepsilon) \frac{\log \log(Kt + T_0)}{\log(Kt + T_0)} \frac{\log T \log t}{(\log \log t)^2} < 0.0069 \frac{\log t}{\log \log t}.$$

Meanwhile, using $\log T = \log t + \log k$ and $\log \log T < \log \log t + 0.02$ ($t \geq H_1$),

$$\begin{aligned} \frac{1}{4\lambda} \int_{-\infty}^{\infty} \frac{\log |\zeta(s_k^* - \lambda^* + iu\delta)|}{(\cosh u)^2} du & \leq \left(\frac{\delta_n E_n}{2} \left(1 + \frac{\log k}{\log H_1} \right) + 0.0069 \right) \frac{\log t}{\log \log t} \\ & \quad + \frac{\log \log t + 0.02 + \log C_n}{2\lambda^*}. \end{aligned} \quad (3.18)$$

On the other hand, we use an estimate of Ford [For02b] (adapted to degree K polynomials) to get

$$-\frac{1}{2} \sum_{k=1}^K a_k \int_{-\infty}^{\infty} \frac{\log |\zeta(s_k^* + \lambda^* + iu\delta)|}{(\cosh u)^2} du \leq a_0 \log \zeta(\sigma^* + \lambda^*). \quad (3.19)$$

Now, applying (3.12) and since $\zeta(\sigma)$ is decreasing for $\sigma > 1$, and $\sigma^* \geq \sigma_1$, one has

$$\frac{\zeta(\sigma^* + \lambda^*)}{2\lambda^*} \leq \frac{\zeta(\sigma_1 + \lambda^*)}{2\lambda^*} \leq \frac{\gamma(\sigma_1 - 1 + \lambda^*) - \log(\sigma_1 - 1 + \lambda^*)}{2\lambda^*}. \quad (3.20)$$

Combining (3.18), (3.19) and (3.20), and substituting the value of σ_1 and the trigonometric polynomial coefficients a_k ,

$$\begin{aligned} & \frac{1}{4\lambda} \sum_{1 \leq k \leq K} a_k \int_{-\infty}^{\infty} \frac{\log |\zeta(s_k^* - \lambda^* + iu\delta)| - \log |\zeta(s_k^* + \lambda^* + iu\delta)|}{(\cosh u)^2} du \\ & \leq (1.7761 \delta_n E_n + 0.0244) \frac{\log t}{\log \log t} + E(t) \end{aligned} \quad (3.21)$$

where

$$E(t) = \frac{3.53(\log \log t + \log C_n) - \log(\lambda^*(t) - 0.004) + 0.07}{2\lambda^*(t)} + \frac{\gamma}{2}.$$

If $n = 3$ then by $\sigma^* < 1$,

$$\log C_n = \frac{14}{3}(\sigma^* - \lambda^*) \log 2.502 + \log 0.0728 < 1.66 - 4.27\lambda^*$$

so that

$$E(t) = \frac{3.53 \log \log t - \log(\lambda^*(t) - 0.004) + 1.73}{2\lambda^*(t)} - 7.24.$$

Substituting $\lambda^* = (8 - E_3 / \log \log t)^{-1}$ and computing derivatives, one finds (with the aid of computer assistance) that for $t \geq H_1$

$$E(t) \frac{\log \log t}{\log t} \leq 0.87153 \dots$$

with the maximum occurring near $t = \exp(111.12 \dots)$.

On the other hand if $n = 4$ then $C_n = 1.549$, so that

$$E(t) = \frac{3.53 \log \log t - \log(\lambda^*(t) - 0.004) + 3.28}{\lambda^*(t)} + \frac{\gamma}{2}.$$

In this case, one finds (with the aid of computer assistance) that for $t \geq T_4$, one has $E(t) \log \log t / \log t \leq 1.0319 \dots$ with the maximum occurring near $t = \exp(222.75 \dots)$. The result follows from substituting these bounds into (3.21). $\square$

Lemma 3.9 (Main term estimate for large t). *Let $t \geq H_2$ and suppose σ^* and λ^* are as in Definition 3.3. If $s_k^* = \sigma^* + ikt$ and $0 < \delta \leq 1/2$ is a constant, then*

$$\begin{aligned} \frac{1}{4\lambda} \sum_{1 \leq k \leq K} a_k \int_{-\infty}^{\infty} \frac{\log |\zeta(s_k^* - \lambda^* + iu\delta)| - \log |\zeta(s_k^* + \lambda^* + iu\delta)|}{(\cosh u)^2} du \\ \leq 3.4431 \frac{\log t}{\log \log t}. \end{aligned}$$

Proof. Once again let $\sigma_n := 1 - n/(2^n - 2)$ (for $n \geq 3$) and $T = kt$ (for $1 \leq k \leq K$). Furthermore let $t_3 = H_2$ and t_n ($n \geq 4$) be defined implicitly via

$$\sigma^*(t_n) - \lambda^*(t_n) = \sigma_n. \tag{3.22}$$

Since $\sigma^*(H_2) - \lambda^*(H_2) = 0.94475 \dots \in [\sigma_3, \sigma_4]$ and $\sigma^*(t) - \lambda^*(t)$ is continuous and decreasing for all $t \geq H_2$, t_n is uniquely defined and increasing in n for all $n \geq 3$.

Now let n be such that $t \in [t_n, t_{n+1})$, so that $\sigma^* - \lambda^* \in [\sigma_n, \sigma_{n+1})$. By Lemma 3.6 and Lemma 3.7, and using $\sigma^* < 1$,

$$\frac{1}{2} \int_{-\infty}^{\infty} \frac{\log |\zeta(s_k^* - \lambda^* + iu\delta)|}{(\cosh u)^2} du < \frac{\lambda^* 2^n - 1}{(n-1)2^n + 2} \log T + \log \log T + \log 1.549.$$

By definition,

$$\frac{\log t}{\log \log t} \frac{\lambda^* 2^n - 1}{(n-1)2^n + 2} = \frac{1}{(n-1)2^n + 2} \left(2^n B \log \log t - \frac{\log t}{\log \log t} \right) = g_n(t), \quad (3.23)$$

say, where $B = 2.248$. Next we show that for all $n \geq 3$,

$$g_n(t) \leq 3.1018 \quad (t_n \leq t \leq t_{n+1}).$$

First suppose that $3 \leq n \leq 7$. Rearranging (3.22) gives

$$\frac{n}{2^n - 2} = 1 - \sigma_n > \lambda^*(t_n) = B \frac{(\log \log t_n)^2}{\log t_n}.$$

By explicitly solving this inequality for t_n (for fixed $3 \leq n \leq 7$), one finds that $\log t_n \geq r_n$, where $r_3 = \log H_2$, $r_4 = 234$, $r_5 = 531$, $r_6 = 1155$ and $r_7 = 2469$. With the aid of computer assistance, we verify that for $3 \leq n \leq 7$,

$$\max_{t_n \leq t \leq t_{n+1}} g_n(t) \leq \max_{t \geq r_n} g_n(t) < 3.1018.$$

Now suppose that $n \geq 8$. The function $h(x) = 2^n B \log x - x / \log x$ is maximized on $(1, \infty)$ at $x = x^*$, where x^* satisfies

$$\frac{2^n B}{x^*} + \frac{1}{(\log x^*)^2} - \frac{1}{\log x^*} = 0.$$

For $n \geq 8$ one has $x^* > 1000$ and so (via a computer verification)

$$\log(2^n B) = \log x^* - \log \log x^* + \log \left(1 - \frac{1}{\log x^*} \right) > \frac{2}{3} \log x^*.$$

Therefore for all $x > 1$,

$$h(x) \leq h(x^*) < 2^n B \log x^* < \frac{3}{2} 2^n B \log(2^n B)$$

and thus, applying the above estimate with $x = \log t$ and by $n \geq 8$,

$$g_n(t) = \frac{2^n B \log x - x / \log x}{(n-1)2^n + 2} < \frac{3B}{2} \frac{2^n(n \log 2 + \log B)}{(n-1)2^n + 2} < 3.1,$$

as required.

Now combining this with

$$\frac{\log T}{\log t} \leq 1 + \frac{\log k}{\log H_1} \quad \text{and} \quad \frac{\log \log T}{\log \log t} \leq 1 + \frac{\log k}{\log H_1 \log \log H_1}$$

one obtains

$$\frac{1}{2} \sum_{1 \leq k \leq K} a_k \int_{-\infty}^{\infty} \frac{\log |\zeta(s_k^* - \lambda^* + iu\delta)|}{(\cosh u)^2} du < 14.48 \log \log t + 1.6. \quad (3.24)$$

For the integral outside the critical strip, we once again follow the argument of [For02b] to get

$$-\frac{1}{2} \sum_{k=1}^K a_k \int_{-\infty}^{\infty} \frac{\log |\zeta(s_k^* + \lambda^* + iu\delta)|}{(\cosh u)^2} du \leq a_0 \log \zeta(\sigma^* + \lambda^*).$$

For $t \geq H_2$, applying (3.12) one has

$$\log \zeta(\sigma^* + \lambda^*) \leq \gamma h(t) - \log h(t), \quad h(t) := B \frac{(\log \log t)^2}{\log t} - A \frac{\log \log t}{\log t}. \quad (3.25)$$

Using $t \geq H_2$ and $0 < A \leq A_1$,

$$(B \log \log H_2 - A_1) \frac{\log \log H_2}{\log t} \leq h(t) < B \frac{(\log \log H_2)^2}{\log H_2}.$$

Substituting the appropriate bound into (3.25) and evaluating constants, one finds $\log \zeta(\sigma^* + \lambda^*) < \log \log t - 4.01$. Combining this with (3.24), and recalling $t \geq H_2$,

$$\begin{aligned} & \frac{1}{4\lambda} \sum_{1 \leq k \leq K} a_k \int_{-\infty}^{\infty} \frac{\log |\zeta(s_k^* - \lambda^* + iu\delta)| - \log |\zeta(s_k^* + \lambda^* + iu\delta)|}{(\cosh u)^2} du \\ & < \frac{15.48 \log \log t}{2B} \frac{\log t}{(\log \log t)^2} < 3.4431 \frac{\log t}{\log \log t}, \end{aligned}$$

as required. $\square$

3.3.2 Bounding the sums over zeroes

In this section, we focus on lower-bounding

$$\sum_{\substack{\rho=\beta+i\gamma \\ \beta \geq \sigma^* - \lambda^*}} \operatorname{Re} F^*(s_k^* - \rho) + \sum_{\substack{\rho=\beta+i\gamma \\ \beta < \sigma^* - \lambda^*}} \operatorname{Re} F_0(s_k^* - \rho),$$

where we recall that F^* and F_0 were defined in Lemma 3.4 and Definition 3.1 respectively.

Although both sums could in theory contain infinitely many zeroes, only those zeroes close to s_k have a significant contribution, since both $F^*(z)$ and $F_0(z)$ decay quickly to 0 as $|z| \rightarrow \infty$. If $k = 1$ then the first sum contains ρ_0 , and our goal is to estimate the contribution of ρ_0 while bounding the contribution of all the other zeroes. By the definition of $F^*(z)$, and since all relevant sums are absolutely convergent, we may rearrange the sums as

$$\begin{aligned} \sum_{\rho: |s_k^* - \rho| \leq \lambda^*} \operatorname{Re} F^*(s_k^* - \rho) + \sum_{\rho: |s_k^* - \rho| > \lambda^*} \operatorname{Re} F_0(s_k^* - \rho) \\ + f(0) \frac{\pi}{2\lambda^*} \sum_{\substack{\rho=\beta+i\gamma \\ \beta \geq \sigma^* - \lambda^* \\ |s_k^* - \rho| > \lambda^*}} \operatorname{Re} \cot \left(\frac{\pi(s_k^* - \rho)}{2\lambda^*} \right). \end{aligned} \quad (3.26)$$

The first sum, containing those zeroes closest to s_k^* , is fortunately easy to bound since we assumed that s_k^* lies sufficiently far inside the zero-free region² so that, for all such zeroes, $\operatorname{Re}(s_k^* - \rho) \geq 0$ and hence $\operatorname{Re} F(s_k^* - \rho) \geq 0$. The majority of this section focuses on bounding the second sum, which produces the main error term. To estimate it we first establish a “local” zero-density estimate to bound the number of zeroes close to s_k^* . We also require an estimate of the total number of non-trivial zeroes in order to bound the last two sums of (3.26).

Definition 3.4 (The quantities $N(T)$ and $N(z, \delta)$). *For any $T > 0$, define $N(T)$ as the number of zeroes ρ of ζ , counted with multiplicity, such that $0 < \operatorname{Im} \rho \leq T$. In addition,*

²Paradoxically, one may have expected bounding this sum would pose the greatest difficulty due to the proximity of zeroes to s_k^* ; indeed it would be difficult to control this sum without assuming that s_k^* lies deep enough inside a known zero-free region.

for any $\delta > 0$ and $z \in \mathbb{C}$, define $N(z, \delta)$ as the number of zeroes ρ of ζ , counted with multiplicity, such that $|z - \rho| \leq \delta$.

The function $N(T)$ is commonly studied in the literature; here we record an explicit result due to Rosser [Ros41].³

Lemma 3.10 (Rosser [Ros41] Theorem 19). *For any $t \geq 2$ one has*

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + \frac{7}{8} + \mathcal{R}(T)$$

where $|\mathcal{R}(T)| \leq \mathcal{R}_0(T) := 0.137 \log T + 0.443 \log \log T + 1.588$.

Meanwhile, $N(z, \delta)$ may be bounded using Lemma 3.3. We follow the approach taken in [For02b, Lemma 4.2], except we use Lemma 3.2 instead of a Vinogradov type estimate.

Recall that $\lambda_2 = 0.29275 \dots$ is a constant defined in (3.11).

Lemma 3.11 (Local zero density estimate). *Let $0 < \delta \leq \lambda_2$ and $z = \sigma + iT$ with $T \geq H_2$ and $1 - \delta/100 \leq \sigma < 1$. Then*

$$N(z, \delta) \leq -\frac{7.6\delta}{\log \delta} \log T + 1.7(\log \log T - \log \delta) + 16.$$

Proof. For convenience, let us write

$$\alpha := 0.333, \quad \beta := 1.529, \quad u := 1 + \alpha\delta + iT, \quad v := \beta\delta.$$

The constants α and β are chosen to approximately minimize the bound on $N(z, \delta)$ when $T = H_2$ and $\delta = \lambda_2$, which are the values most relevant to our application. Note in particular that $\operatorname{Re}(z - \lambda) = 1 + (\alpha - \beta)\delta \geq 1/2$. By Lemma 3.3 with $f = \zeta$, we have

$$\begin{aligned} -2v \operatorname{Re} \frac{\zeta'(u)}{\zeta(u)} &= -\pi \operatorname{Re} \sum_{\substack{\rho = \beta + i\gamma \\ \beta \geq \operatorname{Re}(u-v)}} \cot \left(\frac{\pi(u - \rho)}{2v} \right) \\ &\quad + \frac{1}{2} \int_{-\infty}^{\infty} \frac{\log |\zeta(u - v + 2vix/\pi)| - \log |\zeta(u + v + 2vix/\pi)|}{(\cosh x)^2} dx. \end{aligned}$$

³See also the recent preprint [Bel24] containing a refinement of this result.

Write $\phi := ((\beta - \alpha)\delta \log 2)^{-1}$ for convenience. If we take

$$k = \left\lfloor \frac{\log \phi + \log \log \phi}{\log 2} \right\rfloor \quad (3.27)$$

then (since $\phi > e$)

$$\frac{k}{2^k - 2} > \frac{k}{2^k} \geq \frac{1}{\phi \log 2} \left(1 + \frac{\log \log \phi}{\log \phi} \right) > (\beta - \alpha)\delta$$

i.e. $\operatorname{Re}(u - v) > \sigma_k$. By Lemma 3.7

$$\frac{1}{2} \int_{-\infty}^{\infty} \frac{\log |\zeta(u - v + 2vix/\pi)|}{(\cosh x)^2} dx \leq \frac{1}{2^k - 2} \log t + \log \log t + \log 1.549. \quad (3.28)$$

However (3.27) implies $2^k > (\phi \log \phi)/2 > 2$ so

$$\frac{1}{2^k - 2} < \frac{2}{\phi \log \phi - 4} < -4.576 \frac{\delta}{\log \delta}.$$

Next, for $\sigma > 1$ one has

$$\frac{1}{|\zeta(\sigma + it)|} = \left| \sum_{n \geq 1} \frac{\mu(n)}{n^{\sigma + it}} \right| \leq \sum_{n \geq 1} n^{-\sigma} = \zeta(\sigma)$$

where $\mu(n)$ denotes the Möbius function. By (3.12), $\operatorname{Re}(u + v) = 1 + (\alpha + \beta)\delta$, $\delta \leq \lambda_2$ and $\int_{-\infty}^{\infty} (\cosh x)^{-2} dx = 2$, we get

$$\begin{aligned} -\frac{1}{2} \int_{-\infty}^{\infty} \frac{\log |\zeta(u + v + 2vix/\pi)|}{(\cosh x)^2} dx &\leq \frac{1}{2} \zeta(u + v) \int_{-\infty}^{\infty} \frac{dx}{(\cosh x)^2} \\ &\leq \gamma(\alpha + \beta)\delta - \log(\alpha + \beta)\delta < -0.3 - \log \delta. \end{aligned} \quad (3.29)$$

Also, by [For02b, Lemma 3.1], for $\sigma > 1$ one has $|\zeta'/\zeta(\sigma + it)| \leq (\sigma - 1)^{-1}$ and thus

$$\operatorname{Re} \frac{\zeta'(u)}{\zeta(u)} \leq \left| -\frac{\zeta'(u)}{\zeta(u)} \right| \leq \frac{1}{\operatorname{Re} u - 1} = \frac{1}{\alpha\delta}. \quad (3.30)$$

Next, for real x, y , one has

$$\operatorname{Re} \cot(x + iy) = \frac{\sin 2x}{\cosh 2y - \cos 2x}. \quad (3.31)$$

We use this explicit formula to verify that

$$\operatorname{Re} \cot \left(\frac{\pi \xi}{2\beta} \right) \geq 0.1933 \quad (|\xi - \alpha| \leq 101/100, \operatorname{Re} \xi > \alpha). \quad (3.32)$$

By the maximum modulus principle applied to the function $\xi \mapsto e^{-\cot(\pi\xi/2\beta)}$, it suffices to check that the above inequality holds on the boundary of the semicircular region $\{\xi \in \mathbb{C} : |\xi - \alpha| \leq 1.01, \operatorname{Re} \xi \geq \alpha\}$. Meanwhile, if $|z - \rho| \leq \delta$ then

$$\left| \frac{u - \rho}{\delta} - \alpha \right| - \frac{1}{100} \leq \left| \frac{u - \rho}{\delta} - \alpha - \frac{1 - \sigma}{\delta} \right| \leq 1.$$

Therefore, applying (3.32) termwise with $\xi = (u - \rho)/\delta$,

$$\operatorname{Re} \sum_{\substack{\rho=\beta+i\gamma \\ |z-\rho|\leq\delta}} \cot\left(\frac{\pi(u-\rho)}{2v}\right) \geq 0.1933N(z, \delta).$$

On the other hand, the contribution of the other zeroes is

$$\operatorname{Re} \sum_{\substack{\rho=\beta+i\gamma \\ \beta \geq \operatorname{Re}(u-v) \\ |z-\rho|>\delta}} \cot\left(\frac{\pi(u-\rho)}{2v}\right) \geq 0$$

thanks to (3.31) and $0 < \operatorname{Re}(u - \rho)/v \leq 1$. Combining this with (3.28), (3.29) and (3.30),

$$-\frac{2\beta}{\alpha} + 0.1933\pi N(z, \delta) \leq -\frac{4.576\delta}{\log \delta} \log T + \log \log T + \log 1.549 - 0.3 - \log \delta$$

and the result follows upon rearranging and rounding the constants appropriately. $\square$

Lemma 3.12. *If $T \geq H_1$ then*

$$\sum_{\substack{\rho=\beta+i\gamma \\ |\gamma-T|>1}} |\gamma - T|^{-3} < \log T,$$

where ρ runs through the non-trivial zeroes of the Riemann zeta-function.

Proof. One has

$$\sum_{\substack{\rho=\beta+i\gamma \\ |\gamma-T|>1}} |\gamma - T|^{-3} = \sum_{\rho:\gamma>T+1} \frac{1}{(\gamma - T)^3} + \sum_{\rho:\gamma<T-1} \frac{1}{(T - \gamma)^3}.$$

We estimate the sums using bounds on the zero-counting function $N(T)$. By Lemma 3.10

$$\begin{aligned} \sum_{\rho:\gamma>T+1} \frac{1}{(\gamma - T)^3} &= \int_1^\infty \frac{dN(T+x)}{x^3} \\ &= \frac{(1 - T^{-2}) \log(T+1)}{4\pi} - \frac{\log 2\pi}{4\pi} + \frac{1}{4\pi T} + \int_1^\infty \frac{d\mathcal{R}(T+x)}{x^3} \end{aligned}$$

and, integrating by parts,

$$\int_1^\infty \frac{d\mathcal{R}(T+x)}{x^3} = \left[\frac{\mathcal{R}(T+x)}{x^3} \right]_1^\infty + 3 \int_1^\infty \frac{\mathcal{R}(T+x)}{x^4} dx.$$

Via a routine computation, for $x > 0$ one has $\mathcal{R}_0(T+x) < \mathcal{R}_0(T) + x/T$. As $T \geq H_1$, one has $[\mathcal{R}(T+x)/x^3]_1^\infty < \mathcal{R}_0(T) + 10^{-11}$. In addition,

$$\int_1^\infty \frac{|\mathcal{R}(T+x)|}{x^4} dx < \int_1^\infty \left(\frac{\mathcal{R}_0(T)}{x^4} + \frac{1}{Tx^3} \right) dx \leq \frac{1}{3} \mathcal{R}_0(T) + 10^{-11}.$$

Combining these estimates, one has

$$\sum_{\rho: \gamma > T+1} \frac{1}{(\gamma - T)^3} < \frac{1}{4\pi} \log \frac{T}{2\pi} + 2\mathcal{R}_0(T) + 10^{-10}.$$

Via a similar computation we find (since $\mathcal{R}_0(2) > 0$)

$$\begin{aligned} \sum_{\rho: 2 \leq \gamma < T-1} \frac{1}{(T-\gamma)^3} &= \int_1^{T-2} \frac{dN(T-x)}{x^3} \\ &\leq \frac{\log T}{4\pi} + \left[\frac{\mathcal{R}(T-x)}{x^3} \right]_1^{T-2} + 3 \int_1^{T-2} \frac{|\mathcal{R}(T-x)|}{x^4} dx \\ &< \frac{1}{4\pi} \log \frac{T}{2\pi} + 2\mathcal{R}_0(T) + 10^{-10}. \end{aligned}$$

Also, since the locations of zeroes are symmetric about the real axis,

$$\begin{aligned} \sum_{\rho: \gamma < -2} \frac{1}{(T-\gamma)^3} &= \int_2^\infty \frac{dN(x)}{(T+x)^3} \\ &\leq \frac{1}{2\pi} \int_2^\infty \frac{\log((T+x)/(2\pi))}{(T+x)^3} dx + 3 \int_2^\infty \frac{|\mathcal{R}(x)|}{(T+x)^4} dx < 10^{-10}. \end{aligned}$$

Lastly, since there are no zeroes of $\zeta(s)$ with $|\gamma| \leq 2$, combining the last three inequalities gives (for $T \geq H_1$)

$$\sum_{\substack{\rho=\beta+i\gamma \\ |\gamma-T|>1}} |\gamma-T|^{-3} \leq \frac{1}{2\pi} \log \frac{T}{2\pi} + 4\mathcal{R}_0(T) + 10^{-9} < \log T,$$

as required. □

Lemma 3.13. *If $T \geq H_1$, $z = \sigma + iT$, $\sigma \in [\sigma_1, 1)$ then, for any $0 < \lambda < 1/2$,*

$$\sum_{\substack{\rho=\beta+i\gamma \\ |\gamma-T| \leq 1 \\ |z-\rho| > \lambda}} |z-\rho|^{-3} \leq 0.355(\lambda^{-3} + (2\sigma_1 - 1 - \lambda)^{-3}) \log T - \frac{N(z, \lambda)}{\lambda^3}.$$

Proof. Partition $S := \{\rho : |\operatorname{Im} \rho - T| \leq 1, |z - \rho| > \lambda\}$ into

$$S_1 := \{\rho \in S : |1 - \bar{z} - \rho| > \lambda\},$$

$$S_2 := S \setminus S_1 = \{\rho \in S : |1 - \bar{z} - \rho| \leq \lambda\}.$$

First, suppose $\rho \in S_1$. If ρ lies on the critical line, then $|z - \rho| \geq \sigma_1 - 1/2$ so

$$|z - \rho|^{-3} \leq (\sigma_1 - 1/2)^{-3}.$$

On the other hand, if ρ lies off the critical line, then $1 - \bar{\rho}$ is another zero in S_1 . Since $\operatorname{Re} z \geq \sigma_1$, the total contribution of these zeroes is bounded by

$$|z - \rho|^{-3} + |z - (1 - \bar{\rho})|^{-3} \leq \lambda^{-3} + (2\sigma_1 - 1 - \lambda)^{-3} = 2J(\lambda),$$

say. Since x^{-3} is convex for $x > 0$, by Jensen's inequality one has $J(\lambda) \geq J(\sigma_1 - 1/2) = (\sigma_1 - 1/2)^{-3}$. Thus, in either case one has

$$\sum_{\rho \in S_1} |z - \rho|^{-3} < J(\lambda) |S_1|.$$

The contribution of each zero in S_2 is at most $(2\sigma_1 - 1 - \lambda)^{-3}$, so that since $N(1 - \bar{z}, \lambda) = N(z, \lambda)$,

$$\sum_{\rho \in S_2} |z - \rho|^{-3} \leq (2\sigma_1 - 1 - \lambda)^{-3} N(z, \lambda).$$

It follows from $|S| = |S_1| + 2N(z, \lambda)$ that

$$\sum_{\rho \in S} |z - \rho|^{-3} \leq \frac{\lambda^{-3} + (2\sigma_1 - 1 - \lambda)^{-3}}{2} |S| - \frac{N(z, \lambda)}{\lambda^3}.$$

It remains to bound $|S|$. By Lemma 3.10,

$$\begin{aligned} |S| &< N(T+1) - N(T-1) \\ &\leq \frac{T+1}{2\pi} \log \frac{T+1}{2\pi e} - \frac{T-1}{2\pi} \log \frac{T-1}{2\pi e} + Q_0(T+1) + Q_0(T-1). \end{aligned}$$

If $f(x) := x \log(x/e)$ then $f'(x) = \log x$ is increasing, so

$$\frac{T+1}{2\pi} \log \frac{T+1}{2\pi} - \frac{T-1}{2\pi} \log \frac{T-1}{2\pi} \leq \frac{1}{\pi} \log \frac{T+1}{2\pi} < \frac{1}{\pi} \log \frac{T}{2\pi} + 10^{-10}.$$

Furthermore, since $\log x$ and $\log \log x$ are both concave, we have $\log(T-1) + \log(T+1) \leq 2\log T$ and $\log \log(T-1) + \log \log(T+1) \leq 2\log \log T$ by Jensen's inequality. Therefore, for $T \geq H_1$

$$|S| \leq \frac{1}{\pi} \log \frac{T}{2\pi} + 2Q_0(T) + 10^{-10} < 0.71 \log T. \quad (3.33)$$

The result follows. $\square$

Lemma 3.14. *Let $t \geq H_2$ and suppose $z = \sigma + iT$ with $T \geq t$ and $\sigma_1 \leq \sigma < 1$. Then, for any $0 < \lambda \leq \lambda_2$ one has*

$$\begin{aligned} \sum_{\substack{\rho=\beta+i\gamma \\ |\gamma-T| \leq 1 \\ |z-\rho| > \lambda}} |z-\rho|^{-3} &\leq - \left(140 + \frac{17.09}{\lambda^2 \log \lambda} \right) \log T - 1.7 \left(\frac{\log \lambda}{\lambda^3} + 48 \right) \\ &\quad + (\lambda^{-3} - 39)(1.7 \log \log T + 16) - \frac{N(z, \lambda)}{\lambda^3}. \end{aligned}$$

Note that for the given ranges of λ and T , the right side is always positive.

Proof. We essentially follow the approach taken by Ford [For02b, Lemma 4.3]. Let $S := \{\rho : |\operatorname{Im} \rho - T| \leq 1, |z - \rho| > \lambda\}$ be partitioned into the following sets:

$$\begin{aligned} S_1 &:= \{\rho \in S : |z - \rho| > \lambda_2, |1 - \bar{z} - \rho| > \lambda_2\}, \\ S_2 &:= S \setminus S_1. \end{aligned} \quad (3.34)$$

We estimate the sum over S_1 in the same way as done in the proof of Lemma 3.13. If $\rho \in S_1$ then $1 - \bar{\rho} \in S_1$ so that since $\operatorname{Re} z \geq \sigma_1$ the total contribution of these zeroes is at most $\lambda_2^{-3} + (2\sigma_1 - 1 - \lambda_2)^{-3}$. Thus

$$\sum_{\rho \in S_1} |z - \rho|^{-3} \leq J(\lambda_2) |S_1|, \quad (3.35)$$

where as before $J(x) = x^{-3} + (2\sigma_1 - 1 - x)^{-3}$. Now suppose that $\rho \in S_2$. Then either $|1 - \bar{z} - \rho| \leq \lambda_2$ or $|z - \rho| \leq \lambda_2$, and (by the symmetry of zeroes about the critical line) there are $N(z, \lambda_2)$ zeroes in each case. The zeroes of the former category contribute

$$\sum_{|1 - \bar{z} - \rho| \leq \lambda_2} |z - \rho|^{-3} \leq (2\sigma_1 - 1 - \lambda_2)^{-3} N(z, \lambda_2) \quad (3.36)$$

and, by integrating by parts, the zeroes of the latter category contribute

$$\sum_{|z-\rho|\leq\lambda_2} |z-\rho|^{-3} = \int_{\lambda}^{\lambda_2} \frac{dN(z,u)}{u^3} = \left[\frac{N(z,u)}{u^3} \right]_{\lambda}^{\lambda_2} + 3 \int_{\lambda}^{\lambda_2} \frac{N(z,u)}{u^4} du.$$

Summing the above equation with (3.35), (3.36) and noting that $|S_1| + 2N(z, \lambda_2) = |S|$ gives

$$\sum_{\rho \in S} |z-\rho|^{-3} \leq J(\lambda_2)|S| - \frac{N(z, \lambda)}{\lambda^3} + 3 \int_{\lambda}^{\lambda_2} \frac{N(z, u)}{u^4} du. \quad (3.37)$$

Since $\lambda/(1-\sigma) > 100$ for $t \geq H_2$, we may apply Lemma 3.11 to obtain

$$\begin{aligned} 3 \int_{\lambda}^{\lambda_2} \frac{N(z, u)}{u^4} du &\leq 3 \int_{\lambda}^{\lambda_2} \left(-\frac{7.6 \log T}{u^3 \log u} + \frac{1.7(\log \log T - \log u) + 16}{u^4} \right) du \\ &< -22.8 \log T \int_{\lambda}^{\lambda_2} \frac{du}{u^3 \log u} + 1.7 \left(\frac{\log \lambda}{\lambda^3} + 48 \right) \\ &\quad + (\lambda^{-3} - 39)(1.7 \log \log T + 16). \end{aligned} \quad (3.38)$$

To estimate the remaining integral, first we show

$$\text{Ei}(x) := - \int_{-x}^{\infty} \frac{e^{-t}}{t} dt < \frac{1.5e^x}{x} \quad (x \geq 2)$$

where the integral is interpreted in terms of its Cauchy principal value. The inequality holds for $x \geq 10$ since $\text{Ei}(x) < -\frac{1}{2}e^x \log(1-2/x)$ [AS72, (5.1.20)]⁴, and for $2 \leq x < 10$ we verify that it holds computationally (with room to spare). Therefore

$$\int_{\lambda}^{\lambda_2} \frac{dx}{x^3 \log x} = \text{Ei}(-2 \log \lambda_2) - \text{Ei}(-2 \log \lambda) > 6.86 + \frac{0.75}{\lambda^2 \log \lambda}. \quad (3.39)$$

Lastly, we established in the proof of Lemma 3.13 that $|S| < 0.71 \log T$ so that $J(\lambda_2)|S| < 15.4 \log T$. Combining this with (3.37), (3.38) and (3.39), one obtains the desired result. $\square$

Lemma 3.15. *Suppose $T \geq H_1$ and $1 - \sigma \leq \eta \leq \lambda \leq (\log T)^{-1/2}$. Furthermore suppose that $\zeta(s) \neq 0$ in the rectangle*

$$\{s \in \mathbb{C} : \sigma \leq \text{Re } s \leq 1, |\text{Im } s - T| \leq T_0\}.$$

⁴In [AS72] this inequality is stated in terms of the function $E_1(z) = \int_z^{\infty} t^{-1} e^{-t} dt$, which is related to the exponential integral via $E_1(-x) = -\text{Ei}(x)$ for $x > 0$.

If $z = \sigma + iT$ then

$$f(0) \sum_{\substack{\rho=\beta+i\gamma \\ \beta \geq \sigma-\lambda \\ |z-\rho| > \lambda}} \operatorname{Re} \frac{\pi}{2\lambda} \cot \left(\frac{\pi(z-\rho)}{2\lambda} \right) \geq -10^{-10}\eta.$$

Proof. Recall (3.31), which implies $\operatorname{Re} \cot z \geq 0$ for $0 < \operatorname{Re} z \leq \pi/2$, so

$$\sum_{\substack{\rho=\beta+i\gamma \\ \sigma-\lambda \leq \beta < \sigma \\ |z-\rho| > \lambda}} \operatorname{Re} \cot \left(\frac{\pi(z-\rho)}{2\lambda} \right) \geq 0. \quad (3.40)$$

Now suppose $\rho = \beta + i\gamma$ is a zero with $\beta \geq \sigma$. By assumption, one has $|\gamma - T| > T_0$.

Meanwhile by (3.31), $\cosh 2y > e^{2y}/2$ and $|\sin 2x| \leq |2x|$, one has

$$|\operatorname{Re} \cot(x + iy)| < 10^{-11}cy^{-3} \quad (|x| \leq c, |y| \geq 10^{10}).$$

Since $|\gamma - T| \geq T_0$ and $|\sigma - \beta| < \eta$, we may apply the above bound with $x + iy = \pi(z - \rho)/(2\lambda)$ and $c = \pi\eta/(2\lambda)$ so

$$\left| f(0) \sum_{\substack{\rho=\beta+i\gamma \\ \beta \geq \sigma \\ |z-\rho| > \lambda}} \operatorname{Re} \frac{\pi}{2\lambda} \cot \left(\frac{\pi(z-\rho)}{2\lambda} \right) \right| < 10^{-10}\eta^2\lambda \sum_{\substack{\rho=\beta+i\gamma \\ |\gamma-T| > T_0}} |\gamma - T|^{-3}.$$

By Lemma 3.12 the sum over zeroes is bounded by $\log T$ (with plenty of room to spare),

so

$$\left| f(0) \sum_{\substack{\rho=\beta+i\gamma \\ \beta \geq \sigma \\ |z-\rho| > \lambda}} \operatorname{Re} \frac{\pi}{2\lambda} \cot \left(\frac{\pi(z-\rho)}{2\lambda} \right) \right| < 10^{-10}\eta^2\lambda \log T \leq 10^{-10}\eta.$$

The desired result follows from combining this with (3.40). $\square$

Lemma 3.16. *Let η be as defined in (3.6). Let $\lambda, \nu > 0$ with $\lambda/\eta \geq \nu > 0$ and suppose $z = \sigma + iT$ with $T \geq H_1$ and $1/2 < \sigma < 1$. Suppose further that $\zeta(s) \neq 0$ in the rectangle*

$$\{s \in \mathbb{C} : \sigma \leq \operatorname{Re} s \leq 1, |\operatorname{Im} s - T| \leq \lambda\}. \quad (3.41)$$

If $T = t$ then

$$\operatorname{Re} \sum_{\rho: |z-\rho| \leq \lambda} F^*(z-\rho) \geq F^*(\sigma - \beta_0) - C(0, \nu) \frac{\eta^3}{\lambda^3} N(z, \lambda)$$

and if $T > t$ then

$$\operatorname{Re} \sum_{\rho: |z-\rho| \leq \lambda} F^*(z-\rho) \geq -C(0, \nu) \frac{\eta^3}{\lambda^3} N(z, \lambda).$$

Proof. We will show that for $|z| \leq \lambda$ and $\operatorname{Re} z \geq 0$ one has

$$\operatorname{Re} F^*(z) = \operatorname{Re} F(z) + f(0) \frac{\pi}{2\lambda} \operatorname{Re} \left(\cot \left(\frac{\pi z}{2\lambda} \right) - \frac{2\lambda}{\pi z} \right) \geq -C(0, \nu) \frac{\eta^3}{\lambda^3}. \quad (3.42)$$

Applying the maximum modulus principle to $\exp(-F^*(z))$, it suffices to verify the inequality on the boundary of the region $\{z \in \mathbb{C} : |z| \leq \lambda, \operatorname{Re} z \geq 0\}$. On $\operatorname{Re} z = 0$, one has $\operatorname{Re}(\cot z - 1/z) = 0$ and, by construction, $\operatorname{Re} F(z) \geq 0$ so

$$\operatorname{Re} F^*(z) \geq 0 \quad (\operatorname{Re} z = 0).$$

Meanwhile, on $|z| = \lambda$ and $\operatorname{Re} z \geq 0$ one has

$$\operatorname{Re} F(z) = \operatorname{Re} \frac{f(0)}{z} + \operatorname{Re} F_0(z) = \frac{f(0)}{\lambda^2} \operatorname{Re} z + \operatorname{Re} F_0(z), \quad (3.43)$$

and, by a result due to Ford [For02b, Lemma 4.4] that $\operatorname{Re}(\cot z - 1/z + 4z/\pi^2) \geq 0$, one has

$$f(0) \frac{\pi}{2\lambda} \operatorname{Re} \left(\cot \left(\frac{\pi z}{2\lambda} \right) - \frac{2\lambda}{\pi z} \right) \geq -\frac{f(0)}{\lambda^2} \operatorname{Re} z. \quad (3.44)$$

Since $|z/\eta| = \lambda/\eta \geq \nu$ and $\operatorname{Re}(z/\eta) \geq 0$, one has $|F_0(z)| = |W_0(z/\eta)| \leq C(0, \nu) \eta^3/\lambda^3$ so that, summing (3.43) and (3.44),

$$\operatorname{Re} F^*(z) \geq \operatorname{Re} F_0(z) \geq -C(0, \nu) \eta^3/\lambda^3.$$

Therefore, in either case, (3.42) holds.

By assumption (3.41), all zeroes in the sum

$$\operatorname{Re} \sum_{|z-\rho| \leq \lambda} F^*(z-\rho)$$

satisfy $\operatorname{Re}(z-\rho) \geq 0$. There are $N(z, \lambda)$ zeroes in the sum, so if $T \neq t$ then we apply (3.42) termwise to get

$$\operatorname{Re} \sum_{|z-\rho| \leq \lambda} F^*(z-\rho) \geq -C(0, \nu) \frac{\eta^3}{\lambda^3} N(z, \lambda).$$

If $T = t$ we apply this bound to all zeroes except for $\rho = \rho_0$, to get

$$\operatorname{Re} \sum_{|z-\rho| \leq \lambda} F^*(z - \rho) > F^*(\sigma - \beta_0) - C(0, \nu) \frac{\eta^3}{\lambda^3} N(z, \lambda),$$

as required. $\square$

To summarize, the main conclusion of this section is:

Lemma 3.17. *Suppose Assumption 3.1 holds. If $t \geq H_1$ and $s_k^* = \sigma^* + ikt$ then*

$$\begin{aligned} \sum_{1 \leq k \leq K} a_k & \left(\sum_{\substack{\rho = \beta + i\gamma \\ \beta \leq \sigma^* - \lambda^*}} \operatorname{Re} F_0(s_k^* - \rho) + \sum_{\substack{\rho = \beta + i\gamma \\ \beta > \sigma^* - \lambda^*}} \operatorname{Re} F^*(s_k^* - \rho) \right) \\ & \geq a_1 F^*(\sigma^* - \beta_0) - 0.75\eta \frac{\log t}{(\log \log t)^2}. \end{aligned}$$

Proof. For simplicity, write $z = \sigma^* + iT$ with $t \leq T \leq Kt$. Rearranging sums, we seek a lower bound on

$$\sum_{\substack{\rho \\ |z-\rho| \leq \lambda^*}} \operatorname{Re} F^*(z - \rho) + \sum_{\substack{\rho = \beta + i\gamma \\ |z-\rho| > \lambda^*}} \operatorname{Re} F_0(z - \rho) + f(0) \sum_{\substack{\rho = \beta + i\gamma \\ \beta \geq \sigma^* - \lambda^* \\ |z-\rho| > \lambda^*}} \operatorname{Re} \frac{\pi}{2\lambda^*} \cot \left(\frac{\pi(z - \rho)}{2\lambda^*} \right). \quad (3.45)$$

Consider the terms in the second sum. By Assumption 3.1 one has $\sigma^* > 1 - \eta$ so that $\operatorname{Re}(z - \rho)/\eta > -1$. Furthermore, since $|z - \rho| > \lambda^*$ and $\lambda^*/\eta > \lambda^*/((A + \varepsilon) \log \log t / \log t)$ is increasing in t , one has

$$|(z - \rho)/\eta| \geq \lambda_1/\eta_1 = \nu = 76.36 \dots,$$

say, so that

$$|F_0(z - \rho)| = |W_0((z - \rho)/\eta)| \leq \frac{C(-1, \nu)\eta^3}{|z - \rho|^3}.$$

Therefore

$$\left| \sum_{\substack{\rho = \beta + i\gamma \\ |z-\rho| > \lambda^*}} \operatorname{Re} F_0(z - \rho) \right| \leq C(-1, \nu)\eta^3 \sum_{\substack{\rho = \beta + i\gamma \\ |z-\rho| > \lambda^*}} |z - \rho|^{-3}. \quad (3.46)$$

Note that by $T \leq Kt$, the definition of σ^* and Assumption 3.1, there are no zeroes in the region $|\operatorname{Im} s| \leq T + T_0$ and $\operatorname{Re} s \geq \sigma^*$ so that the zero-free region assumptions of Lemma 3.15 and Lemma 3.16 are satisfied with $\sigma = \sigma^*$.

First, suppose that $H_1 \leq t < H_2$. Applying Lemma 3.12, Lemma 3.14, Lemma 3.15 and Lemma 3.16, the right side of (3.45) is lower-bounded by

$$\mathcal{E}_1 + (C(-1, \nu) - C(0, \nu)) \eta^3 \frac{N(z, \lambda^*)}{(\lambda^*)^3} - 10^{-10} \eta - C(-1, \nu) \eta^3 C_1(T)$$

where $\mathcal{E}_1 = F^*(\sigma - \beta_0)$ if $T = t$ and zero otherwise, and

$$C_1(T) := (0.355(\lambda^{*-3} + (2\sigma_1 - 1 - \lambda^*)^{-3}) + 1) \log T.$$

From Lemma 2.3 one has $C(-1, \nu) > C(0, \nu)$, so the term involving $N(z, \lambda^*)$ may be dropped. Meanwhile, as $t < H_2$ one has by definition $\lambda^* \geq 0.27789\dots$, so $C_1(T) \leq 18.6 \log T$. Furthermore, since $L_1(t) > \log T \geq \log t$ and $L_2(t) < 1.012 \log \log t$ for $t \geq H_1$,

$$C(-1, \nu) \eta^2 C_1(T) \leq 18.6 C(-1, \nu) (A_1 + \varepsilon)^2 \frac{L_2(t)^2}{L_1(t)^2} \log T < 2.56 \frac{(\log \log t)^2}{\log t}.$$

Hence, taking $T = kt$ and summing over k , and since $t \in [H_1, H_2]$,

$$\begin{aligned} \sum_{1 \leq k \leq K} a_k & \left(\sum_{\substack{\rho = \beta + i\gamma \\ \beta \leq \sigma^* - \lambda^*}} \operatorname{Re} F_0(s_k^* - \rho) + \sum_{\substack{\rho = \beta + i\gamma \\ \beta > \sigma^* - \lambda^*}} \operatorname{Re} F^*(s_k^* - \rho) \right) \\ & > a_1 F(\sigma^* - \beta_0) - \eta \sum_{1 \leq k \leq K} a_k \left(2.56 \frac{(\log \log t)^2}{\log t} + 10^{-10} \right) \\ & > a_1 F(\sigma^* - \beta_0) - 0.75 \eta \frac{\log t}{(\log \log t)^2}, \end{aligned}$$

so that the desired result holds for $H_1 \leq t \leq H_2$.

Now consider the case $t \geq H_2$, where from definitions one has $0 < \lambda^* \leq \lambda_2$. This time we apply the same argument, except with Lemma 3.14 in place of Lemma 3.13. The right side of (3.45) is then lower-bounded by

$$\mathcal{E}_1 - 10^{-10} \eta - C(-1, \nu) \eta^3 C_2(\lambda^*, T) \tag{3.47}$$

where

$$\begin{aligned} C_2(\lambda, T) = & - \left(140 + \frac{17.09}{\lambda^2 \log \lambda} \right) \log T - 1.7 \left(\frac{\log \lambda}{\lambda^3} + 48 \right) \\ & + (\lambda^{-3} - 39)(1.7 \log \log T + 16). \end{aligned}$$

Since $\lambda^* = \lambda^*(t) \geq \lambda^*(T)$ and $C_2(\lambda, T)$ is decreasing in λ , the right side is at most $C_2(\lambda^*(T), T)$, which is a pure function of T . We verify, with the aid of *Mathematica*, that

$$C_2(\lambda^*(T), T) \frac{(\log \log T)^4}{(\log T)^3} \leq 1.2532 \dots \quad (T \geq H_2)$$

with the maximum occurring near $\log T = 912.82 \dots$. By Assumption 3.1 and $\log T < 1.013 \log t$ (for $t \geq H_2$ and $T \leq Kt$),

$$\begin{aligned} C(-1, \nu) \eta^2 C_2(\lambda^*, T) &< 1.26 C(-1, \nu) (A_1 + \varepsilon)^2 \frac{(\log \log t)^2}{(\log t)^2} \frac{(\log T)^3}{(\log \log T)^4} \\ &< 0.2 \frac{\log t}{(\log \log t)^2}. \end{aligned}$$

Taking $T = kt$ and summing over k , one obtains

$$\begin{aligned} \sum_{1 \leq k \leq K} a_k \left(\sum_{\substack{\rho = \beta + i\gamma \\ \beta \leq \sigma^* - \lambda^*}} \operatorname{Re} F_0(s_k^* - \rho) + \sum_{\substack{\rho = \beta + i\gamma \\ \beta > \sigma^* - \lambda^*}} \operatorname{Re} F^*(s_k^* - \rho) \right) \\ > a_1 F^*(\sigma^* - \beta_0) - 0.71 \eta \frac{\log t}{(\log \log t)^2}, \end{aligned}$$

so the result follows in this case too. $\square$

3.3.3 Other estimates

In this section we record some lemmas used to estimate the remaining terms of (3.8).

Lemma 3.18. *For any $\eta \in (0, \eta_1]$ and $0 < \lambda < 1/2$, let $F^*(z)$ be defined with respect to η and λ as in Lemma 3.4. If $s_k = \sigma + ikt$ with $\eta/2 < 1 - \sigma < \min(\lambda, \eta)$ and $t \geq H_1$, then*

$$\operatorname{Re} \sum_{1 \leq k \leq K} a_k F^*(s_k - 1) \leq 0.$$

Proof. If $z = s_k - 1$ with $k \geq 1$ then by assumption $|z| \geq H_1$ and $-\eta \leq \operatorname{Re} z \leq -\eta/2$. Then by (3.2)

$$\begin{aligned} \operatorname{Re} F(z) &= \operatorname{Re} W(z/\eta) = \operatorname{Re} \eta w(0)/z + \operatorname{Re} W_0(z/\eta) \\ &\leq (C(-1, H_1/\eta_1) \eta / |z| - w(0)/2) \eta^2 |z|^{-2} < 0. \end{aligned}$$

In addition, if $z = x + iy$ with $-\pi/2 \leq x < 0$ and $y \neq 0$, then

$$\operatorname{Re} \left(\cot z - \frac{1}{z} \right) = \frac{\sin 2x}{\cosh 2y - \cos 2x} \leq 0.$$

It follows that since $-\lambda \leq \operatorname{Re} z < 0$ by assumption,

$$\operatorname{Re} F^*(z) = \operatorname{Re} F(z) + \frac{\pi f(0)}{2\lambda} \operatorname{Re} \left(\cot \left(\frac{\pi z}{2\lambda} \right) - \frac{2\lambda}{\pi z} \right) \leq 0.$$

The result follows from $a_k > 0$. $\square$

Lemma 3.19. *Suppose Assumption 3.1 holds. If $s_k = \sigma + ikt$ with $\sigma \geq \sigma_1$ and $t \geq H_1$, then*

$$\sum_{1 \leq k \leq K} a_k |E(s_k)| \leq 2.9\eta \frac{\log \log t}{\log t}.$$

Proof. We apply Lemma 2.7 from the previous chapter, noting that $t \geq H_1 > H$, $\sigma \geq \sigma_1 \geq \sigma_0$ and (by assumption) $\eta \leq \eta_1 < \eta_0$ so that all conditions are satisfied. One has

$$\sum_{1 \leq k \leq K} a_k |E(s_k)| \leq (14\eta^2 + 425\eta^3) \sum_{1 \leq k \leq K} a_k < 55\eta^2.$$

However Assumption 3.1 implies $\eta \leq (A_1 + \varepsilon) \log \log t / \log t$ so the result follows. $\square$

Lemma 3.20. *Suppose $t \geq H_1$, $\sigma \in [\sigma_1, 1)$ and $0 < \eta \leq (A_1 + \varepsilon) \log \log t / \log t$. If $Q(\sigma)$ is defined in (3.9), then $Q(\sigma) < F(\sigma - 1) - 2.9\eta$.*

Proof. Recall that

$$Q(\sigma) = f(0) \left(-\frac{1}{2} \log \pi + \frac{1}{2} \operatorname{Re} \frac{\Gamma'}{\Gamma} \left(\frac{\sigma}{2} + 1 \right) \right) + F(\sigma - 1) - \operatorname{Re} \sum_{\rho} F(\sigma - \rho) + E(\sigma),$$

where $E(\sigma)$ is defined in Lemma 3.4. Since $F_0(z) = W_0(z/\eta)$, by Lemma 2.7 (in the previous chapter) one has $E(\sigma) \leq 723\eta^3 < 0.01\eta$. In addition, since $\Gamma'/\Gamma(x)$ is increasing for real $x > 1$, and $\sigma < 1$, one has

$$\frac{\Gamma'}{\Gamma} \left(\frac{\sigma}{2} + 1 \right) < \frac{\Gamma'}{\Gamma}(3/2) = 2 - \gamma - \log 4.$$

Lastly, all zeroes $\rho = \beta + i\gamma$ with $|\gamma| \leq H = 3 \cdot 10^{12}$ are known to lie on the critical line [PT21], so $\operatorname{Re}(\sigma - \rho) > 0$ and $\operatorname{Re} F(\sigma - \rho) \geq 0$. On the other hand, if $|\gamma| > H$ then

$|\sigma - \rho|/\eta \geq H/\eta_1$ and $\operatorname{Re}(\sigma - \rho)/\eta \geq -1$, so by (3.2) the total contribution of such zeroes is

$$- \sum_{\substack{\rho=\beta+i\gamma \\ |\gamma|>H}} \operatorname{Re} F(\sigma - \rho) \leq C(-1, H/\eta_1) \eta^3 \sum_{\substack{\rho=\beta+i\gamma \\ |\gamma|>H}} |\sigma - \rho|^{-3} < 46\eta^3 \sum_{\substack{\rho=\beta+i\gamma \\ |\gamma|>H}} |\gamma|^{-3}.$$

By Lehman [Leh66, Lemma 2] with $n = 3$, we have

$$\sum_{\substack{\rho=\beta+i\gamma \\ |\gamma|>H}} |\gamma|^{-3} < \frac{2 \log H}{H^2}$$

and, together with $\eta \leq (A_1 + \varepsilon) \log \log H_1 / \log H_1$, gives (with room to spare)

$$-\operatorname{Re} \sum_{\rho} F(\sigma - \rho) < 10^{-10} \eta.$$

Recalling that $f(0) = \eta w(0)$, the result follows from combining these estimates. $\square$

3.3.4 Putting it all together

We are now ready to prove Proposition 3.1. Recall that for any $\lambda \in (1 - \sigma, 1/2)$ except a set of measure zero, one has (3.8), i.e.

$$0 \leq \sum_{1 \leq k \leq K} a_k I(s_k, \lambda) + a_0 Q(\sigma). \quad (3.48)$$

First suppose that λ^* is not in the exceptional set. Then, we take $\lambda = \lambda^*$, $\sigma = \sigma^*$ and $s_k = s_k^*$. Under Assumption 3.1 and applying Lemma 3.8, Lemma 3.9, Lemma 3.17, Lemma 3.18 and Lemma 3.19, one has

$$\begin{aligned} & \sum_{1 \leq k \leq K} a_k I(s_k^*, \lambda^*) \\ & \leq \frac{f(0)}{4\lambda^*} \sum_{1 \leq k \leq K} a_k \int_{-\infty}^{\infty} \frac{\log |\zeta(s_k^* - \lambda^* + 2\lambda^* iu/\pi)| - \log |\zeta(s_k + \lambda^* + 2\lambda^* iu/\pi)|}{(\cosh u)^2} du \\ & \quad - \sum_{1 \leq k \leq K} a_k \left(\sum_{\substack{\rho=\beta+i\gamma \\ \beta > \sigma^* - \lambda^*}} \operatorname{Re} F^*(s_k^* - \rho) + \sum_{\substack{\rho=\beta+i\gamma \\ \beta \leq \sigma^* - \lambda^*}} \operatorname{Re} F_0(s_k^* - \rho) \right) \\ & \quad + \operatorname{Re} \sum_{1 \leq k \leq K} a_k F^*(s_k^* - 1) + \sum_{1 \leq k \leq K} a_k |E(s_k^*)| \\ & \leq 3.4431 f(0) \frac{\log t}{\log \log t} - a_1 F^*(\sigma^* - \beta_0) + 0.75\eta \frac{\log t}{(\log \log t)^2} + 2.9\eta \frac{\log \log t}{\log t}. \end{aligned}$$

If instead (3.48) does not hold with respect to λ^* , then we simply apply the same argument with a sequence of admissible λ_j that tends to λ^* from below.

Meanwhile, by Lemma 3.20,

$$Q(\sigma^*) \leq F(\sigma^* - 1) - 2.9\eta.$$

Summing these estimates, one obtains

$$0 \leq 3.4431f(0)\frac{\log t}{\log \log t} + a_0F(\sigma^* - 1) - a_1F^*(\sigma^* - \beta_0) + C_3(\eta, t) \quad (3.49)$$

where

$$C_3(\eta, t) := \left(0.75 \frac{\log t}{(\log \log t)^2} + 2.9 \frac{\log \log t}{\log t} - 2.9a_0 \right) \eta.$$

Since the factor in the parentheses is positive for all $t \geq H_1$, we may apply $\eta \leq (A_1 + \varepsilon) \log \log t / \log t$ (from Assumption 3.1) to obtain, after a routine computation, that

$$C_3(\eta, t) \leq 0.00452 \dots \quad (3.50)$$

where the maximum occurs at $t = \exp(1144.7 \dots)$.

Next we estimate the quantity $F^*(\sigma - \beta_0)$. Under Assumption 3.1, for all $t \geq H_1$ we have

$$\frac{\pi(\sigma^* - \beta_0)}{2\lambda^*} < \frac{\pi}{2} \frac{A_1}{\lambda^*(t)} \left(\frac{L_2(t)}{L_1(t)} - \frac{\log \log t}{\log t} \right) < \varepsilon_0 := 7 \cdot 10^{-5}.$$

However for $0 < x \leq \varepsilon_0$ one has $1/x - \cot x \leq \varepsilon_1 := 3 \cdot 10^{-5}$. Therefore

$$F^*(\sigma^* - \beta_0) \geq F(\sigma^* - \beta_0) - \frac{\varepsilon_1 w(0)\eta}{\lambda^*} > F(\sigma^* - \beta_0) - 10^{-6}. \quad (3.51)$$

It remains to estimate the quantity

$$a_1F(\sigma^* - \beta_0) - a_0F(\sigma^* - 1) = \int_0^{2\theta \cot \theta} e^{\mu u} (a_1 e^{-u} - a_0) w(u) du$$

where $\mu := (1 - \sigma^*)/\eta$. Repeating the steps of Lemma 2.14 of the previous chapter (with our new value of θ), the right side is bounded from below by

$$C_4(\mu) := c_0 + c_1\mu + \frac{c_2}{2}\mu^2 + \left(\frac{c_3}{6} - \frac{c^*}{18} \right) \mu^3$$

where

$$c_n := \int_0^{2\theta \cot \theta} u^n (a_1 e^{-u} - a_0) w(u) du, \quad c^* := \int_0^{2\theta \cot \theta} u^3 |a_1 e^{-u} - a_0| w(u) du$$

are computable constants taking the values

$$\begin{aligned} c_0 &= 0.8219901343\dots, & c_1 &= 0.1118643577\dots, & c_2 &= 0.0182350945\dots, \\ c_3 &= -0.0003829770\dots, & c^* &= 0.0188374573\dots \end{aligned}$$

Since $C_4(\mu)$ is increasing on $[\mu_0, 1]$ where $\mu_0 = (1 - \sigma_1)/\eta_1 = 0.96454\dots$,

$$a_1 F(\sigma^* - \beta_0) - a_0 F(\sigma^* - 1) \geq C_4(\mu_0) > 0.9376. \quad (3.52)$$

Therefore, combining (3.49), (3.50), (3.51) and (3.52), one finally has

$$(1 - \beta_0) \frac{\log t}{\log \log t} = \frac{\eta \log t}{\log \log t} > \frac{0.9376 - 10^{-6} a_1 - C_3(\eta, t)}{3.4431 w(0)} \geq 0.05098 > A + \varepsilon,$$

for all $t \geq H_1$, so Proposition 3.1 (and therefore Lemma 3.1) is proved.

Chapter 4

The Vinogradov–Korobov zero-free region

4.1 Introduction

As $t \rightarrow \infty$, the largest known zero-free region for $\zeta(s)$ takes the shape

$$\sigma > 1 - \frac{A}{(\log t)^{2/3}(\log \log t)^{1/3}},$$

for some constant $A > 0$, proved independently by Korobov [Kor58] and Vinogradov [Vin58] in 1958. As in the previous chapter, this zero-free region can be derived by combining Landau’s method with certain non-trivial bounds of $\zeta(s)$ close to the line $\sigma = 1$. In modern treatments, such bounds take the form

$$\zeta(\sigma + it) \ll t^{B(1-\sigma)^{3/2}} (\log t)^{2/3} \quad (1/2 \leq \sigma \leq 1) \quad (4.1)$$

as $t \rightarrow \infty$, where B is an absolute positive constant. These bounds are in turn derived from the deep analysis of certain exponential sums using Vinogradov’s mean value theorem, which are beyond the scope of this chapter. Estimate (4.1) was proved by Richert [Ric67] in 1967 with $B = 100$, and currently the sharpest known estimates of $\zeta(s)$ close to $\sigma = 1$ still takes this form (although substantial progress have been made in reducing the constant

B). Table 4.1 records the historical progression of the constant B and the corresponding improvements in the size of the zero-free region.

Reference	C	B	A
Richert (1967) [Ric67]	–	100 *	–
Heath-Brown (1992) [HB92b]	–	–	$\approx 0.0269B^{-2/3}$ *
Popov (1994) [Pop94]	–	–	0.00006888 *
Kulas (1999) [Kul99]	–	18.497 *	–
Cheng (2000) [Che99; Che00]	175	46	$(952)^{-1}$
Ford (2002) [For02b]	76.2	4.45	$(57.54)^{-1}$
Mossinghoff–Trudigan–Yang (2022) [MTY24]	–	–	$(55.241)^{-1}$
Bellotti (2024) [Bel24]	70.7	4.438	$(53.989)^{-1}$

Table 4.1: Bounds of the form $|\zeta(\sigma + it)| \leq Ct^{B(1-\sigma)^{3/2}}(\log t)^{2/3}$ along with their associated zero-free regions of the form $\sigma > 1 - A(\log t)^{-2/3}(\log \log t)^{-1/3}$ and $t \geq 3$ (entries containing an asterisk (*) indicate that the respective constant only holds for t sufficiently large).

Various methods have been developed to pass from such upper bounds on zeta to a zero-free region, the most successful currently being the zero-detector due to Ford [For02b] that we saw in the previous chapter. Following Ford, all explicit treatments of the Vinogradov–Korobov zero-free region employ this zero-detector, and furthermore each evaluates the zero-detector on the line $\operatorname{Re} s = 1$. As such, an idea explored in the last two chapters, namely, iteratively moving s deeper inside the critical strip, provides a way of refining existing approaches. This is the first idea we shall explore in this chapter.

The second idea we explore is based on the observation that one does not require the full strength of Richert’s bound to prove the Vinogradov–Korobov zero-free region. Instead, we only require (4.1) for σ satisfying

$$1 - \sigma \gg \left(\frac{\log \log t}{\log t} \right)^{2/3}.$$

It turns out that for such σ , one can slightly sharpen (4.1) by a factor of $(\log \log t)^{-1/6}$, i.e.

$$\zeta(\sigma + it) \ll t^{B(1-\sigma)^{3/2}}(\log t)^{2/3}(\log \log t)^{-1/6}, \quad (4.2)$$

where importantly B is the same constant appearing in (4.1). Unfortunately, this estimate generates no improvement in the asymptotic size of the zero-free region as $t \rightarrow \infty$, which depends primarily on the size of B . Nevertheless, we can use this estimate to obtain a

larger zero-free region in certain finite ranges of t , in particular when t is “small”. Our main result of this chapter is

Lemma 4.1. *If $t \geq \exp(544\,000)$ then $\zeta(\sigma + it) \neq 0$ for*

$$\sigma \geq 1 - \frac{1}{51.34} \frac{1}{(\log t)^{2/3} (\log \log t)^{1/3}}.$$

Theorem 1.3 then follows by combining the above lemma with Lemma 2.1 and Lemma 3.1 from the previous chapters (applied in appropriate ranges).

4.2 An explicit estimate of $\zeta(s)$

In this section we derive two explicit estimates of $\zeta(s)$ within certain regions inside the critical strip that are particularly relevant to our application. The first estimate (Lemma 4.2 below) is of the form (4.2) and will be used for large t , and the second estimate (Lemma 4.3) is used for small t .

Throughout we make use of certain explicit exponential sum estimates obtained via the analysis of Vinogradov’s integral, due to Ford [For02a] and Bellotti [Bel24]. We emphasize that the derivations of such bounds is where the main difficulty lies, which are outside the scope of this chapter. Our goal in this section is merely to apply these bounds more efficiently. Let us define

$$S(N, t) := \max_{u \in (0, 1]} \max_{N' \in (N, 2N]} \left| \sum_{N < n \leq N'} (n + u)^{-it} \right|$$

and $\lambda := \log t / \log N$. Ford’s article [For02a] implicitly contains an estimate of the form

$$S(N, t) \leq c_1 e^{c_2 \lambda} N^{1-1/(D_1 \lambda^2)} \quad (N \geq 1) \tag{4.3}$$

for $\lambda \geq 1$, where $c_1 = 2.459$, $c_2 = 1/64.5625$ and $D_1 = 133.66$. This follows from Lemma 6.2 of [For02a] in the range $1 \leq \lambda \leq 2.6$, from Lemma 6.8 and Table 6.1 for $2.6 \leq \lambda \leq 87$, and from Theorem 2 for $\lambda > 87$. Meanwhile, Bellotti [Bel24, Theorem 1.5] has shown that for all $\lambda \geq 1$,

$$S(N, t) \leq c_3 N^{1-1/(D_2 \lambda^2)} \quad (N \geq 1) \tag{4.4}$$

with $c_3 = 8.7979$ and $D_2 = 132.944$.

We apply these bounds to derive two estimates of $\zeta(\sigma + it)$ inside the critical strip (Lemma 4.2 and 4.3 below) which will eventually be used for large and small t respectively. Throughout, let

$$H_3 := \exp(544\,000), \quad H_4 := \exp(4.5 \cdot 10^8).$$

Lemma 4.2 (Estimate for large t). *Let $t \geq 10^{100}$ and $1/2 \leq \sigma \leq 1 - \frac{1}{2}(\log \log t / \log t)^{2/3}$. Then*

$$|\zeta(\sigma + it)| \leq C_1 t^{B(1-\sigma)^{3/2}} (\log t)^{2/3} (\log \log t)^{-1/6}$$

with $C_1 = 85$, $B = 4.438$.

Proof. We begin the same way as Ford [For02a]. By the approximate functional equation¹ (see e.g. Ford [For02a]), for $\sigma \geq 1/2$ and $t \geq 10^{100}$ one has

$$\zeta(\sigma + it) = \sum_{1 \leq n \leq t} n^{-\sigma - it} + O^*(10^{-80}).$$

Let $J = \lceil \log t / \log 2 \rceil - 1$. Dividing the sum into dyadic intervals, applying partial summation and taking $u \rightarrow 0^+$ gives

$$\left| \sum_{1 \leq n \leq t} n^{-\sigma - it} \right| \leq 1 + \sum_{j=0}^J \left| \sum_{2^j < n \leq \min(2^{j+1}, t)} n^{-\sigma - it} \right| \leq 1 + \sum_{j=0}^J (2^j)^{-\sigma} S(2^j, t).$$

Applying the estimate (4.4) to each term in the sum gives

$$\left| \sum_{1 \leq n \leq t} n^{-\sigma - it} \right| \leq 1 + c_3 \sum_{j=0}^J \exp(h(j \log 2)), \quad (4.5)$$

where

$$h(x) := (1 - \sigma)x - \frac{x^3}{D_2(\log t)^2}. \quad (4.6)$$

¹One could also use the Riemann–Siegel formula instead of Euler–Maclaurin summation here, to obtain an approximate functional equation for $\zeta(s)$ involving shorter Dirichlet polynomials of length $O(t^{1/2})$. However, in our application the dominant terms in Dirichlet polynomial occur when n is rather small (specifically $n \asymp (\log t)^{1/3+o(1)}$) and the savings from shortening the polynomial are negligible.

For $x \geq 0$, the function h is unimodal, concave down and is maximized at $x = x_0 := \sqrt{D_2(1-\sigma)/3} \log t$, so

$$\sum_{0 \leq j \leq J} \exp(h(j \log 2)) \leq e^{h(x_0 \log 2)} + \int_0^\infty e^{h(u \log 2)} du. \quad (4.7)$$

The first term on the right side is $t^2 \sqrt{D_2/27} (1-\sigma)^{3/2}$. To estimate the integral we use an explicit one-dimensional saddle-point approximation. Applying the substitution

$$y = \sqrt{\frac{1-\sigma}{3}} D_2^{1/6} (\log t)^{1/3}, \quad \alpha = \frac{\log 2}{y D_2^{1/3} (\log t)^{2/3}} u,$$

one obtains

$$\int_0^\infty e^{h(u \log 2)} du = \frac{y D_2^{1/3} (\log t)^{2/3}}{\log 2} \int_0^\infty e^{(3\alpha - \alpha^3)y^3} d\alpha. \quad (4.8)$$

Since $3\alpha - \alpha^3 \leq 2 - 2(\alpha - 1)^2$ for all $\alpha \geq 0$, one has

$$e^{-2y^3} \int_0^\infty e^{(3\alpha - \alpha^3)y^3} d\alpha \leq \int_0^\infty e^{-2y^3(\alpha - 1)^2} d\alpha < \int_{-\infty}^\infty e^{-2y^3(\alpha - 1)^2} d\alpha = \sqrt{\frac{\pi}{2y^3}}. \quad (4.9)$$

Since $\sigma \leq 1 - \frac{1}{2}(\log \log t / \log t)^{2/3}$ by assumption, one has

$$y \geq \frac{D_2^{1/6}}{\sqrt{6}} (\log \log t)^{1/3}. \quad (4.10)$$

Writing $B = 2(D_2/27)^{1/2}$ and combining (4.5), (4.7), (4.8), (4.9) and (4.10), for $1/2 \leq \sigma \leq 1 - \frac{1}{2}(\log \log t / \log t)^{2/3}$ one has

$$\begin{aligned} |\zeta(\sigma + it)| &\leq 1 + 10^{-80} + c_3 t^{B(1-\sigma)^{3/2}} + c_3 \sqrt{\frac{\pi}{2}} \frac{(6D_2)^{1/4} t^{B(1-\sigma)^{3/2}} (\log t)^{2/3}}{\log 2 (\log \log t)^{1/6}} \\ &\leq 85 t^{4.438(1-\sigma)^{3/2}} (\log t)^{2/3} (\log \log t)^{-1/6} \quad (t \geq 10^{100}), \end{aligned}$$

as required. $\square$

Lemma 4.3 (Estimate for small t). *Suppose $H_3/2 \leq t \leq H_4$ and $1/2 \leq \sigma \leq 1 - \frac{1}{2}(\log \log t / \log t)^{2/3}$. Then*

$$|\zeta(\sigma + it)| \leq C_2 t^{B_2(\phi)(1-\sigma)^{3/2}} (\log t)^{2/3} (\log \log t)^{-1/6}$$

where $C_2 = 25$, $B_2(\phi) = 4.45(1 + 3\phi/2 + \phi^2/2)$ and $\phi = (2876(1-\sigma)^2 \log t)^{-1}$.

Proof. As before we seek to bound $S(N, t)$ for $1 \leq N \leq t$. Throughout let

$$y = \sqrt{\frac{1-\sigma}{3}} D_1^{1/6} (\log t)^{1/3}, \quad \alpha = \frac{\log N}{y D_1^{1/3} (\log t)^{2/3}}$$

and recall that $\lambda = \log t / \log N$. Suppose first that

$$\lambda < 2\sqrt{\frac{3}{D_1(1-\sigma)}}$$

so that $\alpha > \alpha_0 = 1/2$. Via (4.3) and a routine calculation one obtains

$$N^{-\sigma} S(N, t) \leq c_1 \exp(g(\alpha)y^3), \quad g(\alpha) = 3\frac{\phi}{\alpha} + 3\alpha - \alpha^3.$$

By explicitly computing derivatives, we find that $g(\alpha)$ is locally maximized at the point

$$\alpha^* = \left(\frac{1 + \sqrt{1-4\phi}}{2} \right)^{1/2}.$$

Taking a Taylor expansion at α^* , for each $\alpha \geq \alpha_0$ there exists $\xi = \xi(\alpha) \geq \alpha_0$ such that

$$g(\alpha) = g(\alpha^*) + g'(\alpha^*)(\alpha - \alpha^*) + \frac{1}{2}g''(\alpha^*)(\alpha - \alpha^*)^2 + \frac{1}{6}g'''(\xi)(\alpha - \alpha^*)^3.$$

Note that $g'(\alpha^*) = 0$. Additionally, a routine computation reveals that for relevant σ and t one has $0 < \phi \leq \phi_0 = 0.027$, so that

$$g(\alpha^*) = \frac{\sqrt{2}(1 + 4\phi + \sqrt{1-4\phi})}{(1 + \sqrt{1-4\phi})^{1/2}} < 2 + 3\phi + \phi^2,$$

$$g''(\alpha^*) = -6\sqrt{2} \left(\frac{1-4\phi}{1 + \sqrt{1-4\phi}} \right)^{1/2} < -5.81.$$

Furthermore, $g'''(\xi) = -6 - 18\phi/\xi^4 < 0$. If $\alpha \geq \alpha^*$ then $g'''(\xi)(\alpha - \alpha^*)^3 \leq 0$ and if $\alpha_0 \leq \alpha \leq \alpha^*$ then $|\alpha - \alpha^*| < 1 - \alpha_0$, so in either case

$$\frac{1}{6}g'''(\xi)(\alpha - \alpha^*)^3 < (1 + 3\phi_0\alpha_0^{-4})(1 - \alpha_0)(\alpha - \alpha^*)^2 < (\alpha - \alpha^*)^2.$$

We conclude that

$$N^{-\sigma} S(N, t) \leq c_1 \exp(g(\alpha)y^3) \leq c_1 \exp((2 + 3\phi + \phi^2 - 1.9(\alpha - \alpha^*)^2)y^3). \quad (4.11)$$

Now suppose that

$$\lambda \geq 2\sqrt{\frac{3}{D_1(1-\sigma)}},$$

so that $\alpha \leq \alpha_0$. In this case, we apply (4.3) to obtain

$$N^{-\sigma} S(N, t) \leq c_1 \exp \left(\log \frac{c_3}{c_1} + \left(3\alpha - \frac{D_2}{D_1} \alpha^3 \right) y^3 \right). \quad (4.12)$$

For any $0 < \delta \leq 1$ and $\alpha^* < 1$, the function $3\alpha - \delta\alpha^3 + 1.9(\alpha - \alpha^*)^2$ is increasing for $\alpha \in [0, 1]$. Therefore, for $0 \leq \alpha < \alpha_0 < \alpha^* < 1$ one has

$$3\alpha - \frac{D_2}{D_1} \alpha^3 + 1.9(\alpha - \alpha^*)^2 < 3\alpha_0 - \frac{D_2}{D_1} \alpha_0^3 + 1.9(\alpha_0 - 1)^2 < 1.87. \quad (4.13)$$

In addition, since $\sigma \leq 1 - \frac{1}{2}(\log \log t / \log t)^{2/3}$ by assumption, we have

$$y \geq \frac{D_1^{1/6}}{\sqrt{6}} (\log \log t)^{1/3} \quad (4.14)$$

so that for $t \geq H_3$

$$y^{-3} \log \frac{c_3}{c_1} < 0.13. \quad (4.15)$$

Thus by (4.12), (4.13) and (4.15), we have

$$N^{-\sigma} S(N, t) \leq c_1 \exp((2 - 1.9(\alpha - \alpha^*)^2) y^3) \quad (4.16)$$

in this case.

Combining (4.11) and (4.16), one has for any $1 \leq N \leq t$ that

$$N^{-\sigma} S(N, t) \leq c_1 e^{(2+3\phi+\phi^2-1.9(\alpha-\alpha^*)^2)y^3}.$$

Applying this for $N = j \log 2$ for $0 \leq j \leq J = \lceil \log t / \log 2 \rceil - 1$,

$$\left| \sum_{1 \leq n \leq t} n^{-\sigma-it} \right| \leq 1 + \sum_{j=0}^J (2^j)^{-\sigma} S(2^j, t) \leq 1 + c_1 e^{(2+3\phi+\phi^2)y^3} \sum_{j=0}^J e^{-1.9(\alpha_j - \alpha^*)^2 y^3}$$

where α_j is α evaluated at $N = j \log 2$. Proceeding as in the previous lemma, we have

$$\begin{aligned} \sum_{j=0}^J e^{-1.9(\alpha_j - \alpha^*)^2 y^3} &\leq 1 + \frac{y D_1^{1/3} (\log t)^{2/3}}{\log 2} \int_0^J e^{-1.9(\alpha - \alpha^*)^2 y^3} d\alpha \\ &< 1 + \frac{D_1^{1/3} (\log t)^{2/3}}{\log 2} \sqrt{\frac{\pi}{1.9y}}. \end{aligned}$$

Substituting and applying (4.14), we have

$$\begin{aligned} |\zeta(\sigma + it)| &< 1 + 10^{-80} + c_1 \left(1 + \left(\frac{\pi}{1.9} \right)^{1/2} \frac{(6D_1)^{1/4}}{\log 2} \frac{(\log t)^{2/3}}{(\log \log t)^{1/6}} \right) e^{(2+3\phi+\phi^2)y^3} \\ &\leq 25 t^{(2+3\phi+\phi^2)\sqrt{D_1/27}(1-\sigma)^{3/2}} (\log t)^{2/3} (\log \log t)^{-1/6}. \end{aligned}$$

The result follows. $\square$

4.3 Proof of Lemma 4.1

In this section we prove Lemma 4.1. The approach is similar to that of Lemma 3.1 from the previous chapter, so we shall reuse results where possible. Throughout, let

$$A_2 := (51.34)^{-1} = 0.019477\dots$$

In the spirit of the inductive argument used in the previous chapters, let us define, for any $A \in (0, A_2]$, the hypothesis

$$\mathcal{H}_1(A) : \quad \zeta(\sigma + it) \neq 0 \quad \text{for} \quad \sigma > 1 - \frac{A}{(\log t)^{2/3}(\log \log t)^{1/3}}, \quad t \geq H_3.$$

Vinogradov [Vin58] and Korobov [Kor58] showed that $\mathcal{H}_1(A)$ holds for some $A > 0$. It then suffices to show that $\mathcal{H}_1(A) \implies \mathcal{H}_1(A + \varepsilon)$ for all $0 < A \leq A_2$ where as before $\varepsilon = 10^{-100}$ is a sufficiently small constant. We show this via a contradiction to the following assumption.

Assumption 4.1. *For some $0 < A \leq A_2$, hypothesis $\mathcal{H}_1(A)$ holds. Furthermore, hypothesis $\mathcal{H}_1(A + \varepsilon)$ does not hold, i.e. there exists a zero $\rho_0 = \beta_0 + it$ with $t \geq H_3$ and*

$$A \leq (1 - \beta_0)(\log t)^{2/3}(\log \log t)^{1/3} < A + \varepsilon.$$

Throughout, we find it once again convenient to write

$$\eta := 1 - \beta_0.$$

We start by specifying our choices of a smoothing function f and a non-negative trigonometric polynomial P . Compared to previous chapters, here we choose a trigonometric polynomial of larger degree, found in [MTY24]:

$$P(x) = \sum_{0 \leq k \leq K} b_k \cos kx \tag{4.17}$$

where $K = 40$ and the coefficients b_k are recorded in [MTY24, Table 2]. In particular, $b_k > 0$ and

$$b_0 = 1, \quad b_1 = 1.74600190914994, \quad b = \sum_{1 \leq k \leq K} b_k = 3.56453965437134. \tag{4.18}$$

On the other hand we retain the same choice of a smoothing function f as the previous chapter (in Definition 3.2), albeit with a different choice of θ . Here, we take θ to be the unique solution of the equation

$$\sin^2 \theta = \frac{b_1}{b_0} (1 - \theta \cot \theta) \quad (0 < \theta < \pi/2). \quad (4.19)$$

Following the discussion in [HB92a], one expects this choice to be optimal as $t \rightarrow \infty$, and the range of t we encounter in this chapter are large enough to support this choice. For the values of b_0, b_1 in (4.18) we may explicitly compute

$$\theta = 1.13269369969232 \dots$$

Equipped with these choices, we follow the argument of the previous chapter. For each fixed σ and all $1 - \sigma < \lambda < 1/2$ except for a set of Lebesgue measure zero, one has

$$0 \leq \sum_{n \geq 1} \frac{\Lambda(n)}{n^\sigma} f(\log n) P(t \log n) \leq \sum_{1 \leq k \leq K} b_k I(\sigma + ikt, \lambda) + Q(\sigma), \quad (4.20)$$

where I and Q are defined in Lemma 3.4 and (3.9) respectively. We seek an upper bound on the terms appearing on the right side; this is our objective throughout the rest of this section. Lemma 4.2 and Lemma 4.3 play prominent roles in this estimate, and we recall for convenience the constants

$$B = 4.438, \quad C_1 = 85, \quad C_2 = 25.$$

We choose λ relative to the constant B ; specifically, we take $\lambda = \lambda^*$ where²

$$\lambda^* := \lambda^*(t) = EB^{-2/3} \left(\frac{\log \log t}{\log t} \right)^{2/3}, \quad E = 1.63. \quad (4.21)$$

In addition, we choose $\sigma = \sigma^*$, defined as

$$\sigma^* := \sigma^*(t) = 1 - \frac{A}{L_1^{2/3} L_2^{1/3}}, \quad (4.22)$$

²The constant $E = 1.63$ replaces $(4/3(1 + b_0/b))^{2/3} = 1.428 \dots$ appearing in Ford's treatment, and determines how we weight the main and secondary terms appearing in the estimate in Lemma 4.4. As $t \rightarrow \infty$, Ford's choice is optimal; however for finite t it is more favourable to take E slightly larger.

where

$$L_1(t) := \log(Kt + T_0), \quad L_2 := \log \log(Kt + T_0), \quad T_0 := 10^{10}.$$

We also define the constants

$$\begin{aligned} \lambda_3 &:= EB^{-2/3} \left(\frac{\log \log H_3}{\log H_3} \right)^{2/3} \approx 5.06 \cdot 10^{-4}, \\ \sigma_3 &:= 1 - \frac{A_2}{L_1(H_3)^{2/3} L_2(H_3)^{1/3}} \approx 1 - 1.24 \cdot 10^{-6}, \end{aligned}$$

so that $\lambda^* \in (0, \lambda_3]$ and $\sigma^* \in [\sigma_3, 1)$ for all $t \geq H_3$.

In the next few lemmas we repeat the arguments of the previous chapter to derive estimates of I and Q in our new setting. First, Lemma 4.2 gives

Lemma 4.4 (Main term estimate). *Let $t \geq H_3$ and let $\sigma^*(t)$, $\lambda^*(t)$ be as defined above.*

If $s_k^ := \sigma^* + ikt$, then for any fixed $0 < \delta \leq 1/2$,*

$$\begin{aligned} & \frac{1}{4\lambda^*} \sum_{1 \leq k \leq K} a_k \int_{-\infty}^{\infty} \frac{\log |\zeta(s_k^* - \lambda^* + iu\delta)| - \log |\zeta(s_k^* + \lambda^* + iu\delta)|}{(\cosh u)^2} du \\ & \leq 9.108(\log t)^{2/3}(\log \log t)^{1/3} + 1.263 \frac{(\log t)^{2/3}}{(\log \log t)^{2/3}}. \end{aligned}$$

Proof. Suppose throughout that $t \geq H_3$ and $y \geq t/2$. By inspection, one has

$$\sigma^*(t) - \lambda^*(t) < 1 - \frac{1}{2} \left(\frac{\log \log(t/3)}{\log(t/3)} \right)^{2/3} \leq 1 - \frac{1}{2} \left(\frac{\log \log y}{\log y} \right)^{2/3}. \quad (4.23)$$

Suppose first that $t \geq H_4^X$ where $X := 1/10$. In light of (4.23) we may apply Lemma 4.2 to obtain

$$\begin{aligned} \log |\zeta(\sigma^* - \lambda^* + iy)| &< B(\lambda^*)^{3/2} \log y + \frac{2}{3} \log \log y - \frac{1}{6} \log \log \log y + \log C_1 \\ &\leq B(\lambda^*)^{3/2} \log y + \frac{2}{3} \log \log y + 3.965 \end{aligned} \quad (4.24)$$

since $y \geq H_4^X/2$. Now let $T \in [t, Kt]$. By Lemma 3.7 and the inequality $\log T \leq (1 + \log K / \log H_4) \log t$, one has

$$\begin{aligned} \frac{1}{2} \int_{-\infty}^{\infty} \frac{\log |\zeta(\sigma^* - \lambda^* + i(u\delta + T))|}{(\cosh u)^2} du &< B(\lambda^*)^{3/2} \log T + \frac{2}{3} \log \log T + 3.965 \\ &< 2.897 \log \log T + 1 \quad (t \geq H_4^X). \end{aligned} \quad (4.25)$$

Now suppose that $H_3 \leq t < H_4^X$. If $t/2 \leq y \leq t^{1/X}$ then by Lemma 4.3,

$$\begin{aligned} \log |\zeta(\sigma^* - \lambda^* + iy)| &< B_2(\phi^*)(\lambda^*)^{3/2} \log y + \frac{2}{3} \log \log y \\ &\quad - \frac{1}{6} \log \log \log y + \log C_2 \end{aligned}$$

where $\phi^* := (R(\lambda^*)^2 \log t)^{-1}$ and $R = 2876$. However the same bound also holds for $y > t^{1/X}$ thanks to Lemma 4.2, since in this range

$$\begin{aligned} (B_2(\phi^*) - B)(\lambda^*)^{3/2} \log y &> \frac{E^{3/2}(4.45 - B)}{XB} \log \log t + \frac{3B^{4/3}}{2XRE^{1/2}} \frac{(\log t)^{1/3}}{(\log \log t)^{1/3}} \\ &\quad + \frac{B^{8/3}}{2XR^2E^{5/2}} \frac{(\log t)^{2/3}}{(\log \log t)^{5/3}} \\ &> \log(C_1/C_2), \end{aligned}$$

where the last inequality was verified with computer assistance.

Therefore, for all $T \in [t, Kt]$ and by Lemma 3.7,

$$\frac{1}{2} \int_{-\infty}^{\infty} \frac{\log |\zeta(\sigma^* - \lambda^* + i(u\delta + T))|}{(\cosh u)^2} du < 4.45(\lambda^*)^{3/2} \log T + \frac{2}{3} \log \log T + \mathcal{E}(t, T)$$

where

$$\begin{aligned} \mathcal{E}(t, T) &:= (B_2(\phi^*) - 4.45) \log T - \frac{1}{6} \log \log \log T + \log C_2 \\ &< 0.004 \frac{(\log T)^{1/3}}{(\log \log T)^{1/3}} + 10^{-6} \frac{(\log T)^{2/3}}{(\log \log T)^{5/3}} - \frac{1}{6} \log \log \log T + \log C_2. \end{aligned}$$

Substituting the definition of λ^* and applying $H_3 \leq t \leq T \leq Kt < KH_4^X$ yields

$$\frac{1}{2} \int_{-\infty}^{\infty} \frac{\log |\zeta(\sigma^* - \lambda^* + i(u\delta + T))|}{(\cosh u)^2} du < 2.897 \log \log T + 1 \quad (H_3 \leq t < H_4^X). \quad (4.26)$$

Combining the cases (4.25) and (4.26), and taking $T = kt$ ($1 \leq k \leq K$), one obtains

$$\frac{1}{4\lambda^*} \sum_{1 \leq k \leq K} b_k \int_{-\infty}^{\infty} \frac{\log |\zeta(s_k^* - \lambda^* + iu\delta)|}{(\cosh u)^2} du < 8.555(\log t)^{2/3}(\log \log t)^{1/3} + \frac{b}{2\lambda^*} \quad (4.27)$$

for all $t \geq H_3$.

Meanwhile, as in the previous chapter, since $\sigma^* + \lambda^* > 1$ we apply [For02b, Lemma 5.1]

(generalized in the usual way to degree K polynomials) to obtain

$$-\frac{1}{4\lambda^*} \sum_{1 \leq k \leq K} b_k \int_{-\infty}^{\infty} \frac{\log |\zeta(s_k^* + \lambda^* + iu\delta)|}{(\cosh u)^2} du \leq b_0 \frac{\log \zeta(\sigma^* + \lambda^*)}{2\lambda^*}. \quad (4.28)$$

Applying the estimate (3.12),

$$\begin{aligned} \log \zeta(\sigma^* + \lambda^*) &\leq \gamma(\sigma^* + \lambda^* - 1) - \log(\sigma^* + \lambda^* - 1) \\ &< \gamma\lambda^* - \log \lambda^* - \log \left(1 - \frac{1 - \sigma^*}{\lambda^*}\right). \end{aligned}$$

Since $(1 - \sigma^*)/\lambda^* < 0.003$ for $t \geq H_3$, and substituting the definition of λ^* , one has

$$\begin{aligned} \frac{\log \zeta(\sigma^* + \lambda^*)}{2\lambda^*} &< \frac{1}{2\lambda^*} \left(\frac{2}{3} \log \log t - \frac{2}{3} \log \frac{B^{2/3} \log \log H_3}{E} - \log 0.997 \right) + \frac{\gamma}{2} \\ &< \frac{B^{2/3}}{3E} (\log t)^{2/3} (\log \log t)^{1/3} - \frac{1.02}{\lambda^*} + \frac{\gamma}{2}. \end{aligned}$$

Combining this with (4.27) and (4.28) gives

$$\begin{aligned} \frac{1}{4\lambda^*} \sum_{1 \leq k \leq K} b_k \int_{-\infty}^{\infty} \frac{\log |\zeta(s_k^* - \lambda^* + iu\delta)| + \log |\zeta(s_k^* + \lambda^* + iu\delta)|}{(\cosh u)^2} du \\ &< 9.108 (\log t)^{2/3} (\log \log t)^{1/3} + \frac{b/2 - 1.02}{\lambda^*} + \frac{\gamma}{2} \\ &< 9.108 (\log t)^{2/3} (\log \log t)^{1/3} + 1.263 \frac{(\log t)^{2/3}}{(\log \log t)^{2/3}}, \end{aligned}$$

for all $t \geq H_3$, as required. $\square$

Next we establish a bound on $N(z, \delta)$, which we recall is the number of zeroes ρ satisfying $|z - \rho| \leq \delta$. This lemma serves a similar purpose as Lemma 3.10 (of the previous chapter) but is sharper for values of δ, T most relevant to our application.

Lemma 4.5 (Local zero density estimate). *Let $0 < \delta \leq 1/4$ and $z = \sigma + iT$ with $T \geq H_3$ and $\sigma \in [1 - \delta/100, 1)$. Then*

$$N(z, \delta) \leq 6.1 \delta^{3/2} \log T + 0.36 \log \log T - 0.53 \log \delta + 6.$$

Proof. Applying Lemma 3.3 with $s = u$ and $\lambda = v$ and $f = \zeta$, one has

$$\begin{aligned} -2v \operatorname{Re} \frac{\zeta'(u)}{\zeta(u)} &= -\pi \operatorname{Re} \sum_{\substack{\rho = \beta + i\gamma \\ \beta \geq \operatorname{Re}(u-v)}} \cot \left(\frac{\pi(u - \rho)}{2v} \right) \\ &\quad + \frac{1}{2} \int_{-\infty}^{\infty} \frac{\log |\zeta(u - v + 2vix/\pi)| - \log |\zeta(u + v + 2vix/\pi)|}{(\cosh x)^2} dx. \end{aligned}$$

We take $v = 1 + \alpha_1\delta + iT$, $u = \alpha_2\delta$ where $\alpha_1 = 0.652$, $\alpha_2 = 2.541$ (these constants are chosen to optimize the bound on $N(z, \delta)$ when $\delta = \lambda_3$ and $T = H_3$). By Lemma 3.7 and [Bel24, Theorem 1.1],

$$\frac{1}{2} \int_{-\infty}^{\infty} \frac{\log |\zeta(u - v + 2vix/\pi)|}{(\cosh x)^2} dx \leq B((\alpha_2 - \alpha_1)\delta)^{3/2} \log T + \frac{2}{3} \log \log T + \log C_0$$

where $C_0 = 70.7$. In addition, following the same argument leading up to (3.29), (3.30) and (3.32) respectively in the previous chapter, one has the series of estimates

$$-\frac{1}{2} \int_{-\infty}^{\infty} \frac{\log |\zeta(u + v + 2vix/\pi)|}{(\cosh x)^2} dx \leq \gamma(\alpha_1 + \alpha_2)\delta - \log(\alpha_1 + \alpha_2)\delta,$$

$$2v \operatorname{Re} \frac{\zeta'(u)}{\zeta(u)} \leq \frac{2\alpha_2}{\alpha_1},$$

$$\operatorname{Re} \cot \left(\frac{\pi \xi}{2\alpha_2} \right) \geq 0.6041 \quad (|\xi - \alpha_1| \leq 101/100, \operatorname{Re} \xi > \alpha_1).$$

Combining these estimates gives

$$\begin{aligned} -\frac{2\alpha_2}{\alpha_1} + 0.6041\pi N(z, \delta) &\leq B(\alpha_2 - \alpha_1)^{3/2} \delta^{3/2} \log T + \frac{2}{3} \log \log T + \log C_0 \\ &\quad + \gamma(\alpha_1 + \alpha_2)\delta - \log(\alpha_1 + \alpha_2) - \log \delta. \end{aligned}$$

The desired result follows from substituting the values of α_1 , α_2 , B , C_0 and applying $\delta \leq 1/4$. $\square$

The following two lemmas make use of this density bound to estimate the contribution of various sums over zeroes appearing in $I(s_k, \lambda)$.

Lemma 4.6. *If $t \geq H_3$ and $z = \sigma + iT$ with $\sigma \in [\sigma_3, 1)$ and $T \in [t, Kt]$ then*

$$\sum_{\substack{\rho = \beta + i\gamma \\ |z - \rho| > \lambda^*}} |z - \rho|^{-3} \leq 28.3 \frac{(\log T)^2}{\log \log T} - \frac{N(z, \lambda^*)}{(\lambda^*)^3}.$$

Proof. This result is conceptually similar to Lemma 3.14 of the previous chapter. We once

again follow the argument in Ford [For02b], by dividing the zeroes in the sum into the sets

$$\begin{aligned} S_1 &= \{\rho : |\operatorname{Im}\rho - T| > 1\}, \\ S_2 &= \{\rho : |z - \rho| > \delta, |1 - \bar{z} - \rho| > \delta, |t - \operatorname{Im}\rho| \leq 1\}, \\ S_3 &= \{\rho : |1 - \bar{z} - \rho| \leq \delta, |t - \operatorname{Im}\rho| \leq 1\}, \\ S_4 &= \{\rho : \lambda^* < |z - \rho| \leq \delta : |t - \operatorname{Im}\rho| \leq 1\} \end{aligned}$$

for some constant $\delta \in (\lambda^*, 1/2)$ to be chosen later. From Lemma 3.12 (of the previous chapter), the zeroes in S_1 contribute

$$\sum_{\rho \in S_1} |z - \rho|^{-3} < \log T.$$

Meanwhile, each pair of zeroes $\rho, 1 - \bar{\rho}$ in S_2 contributes

$$|z - \rho|^{-3} + |z - (1 - \bar{\rho})|^{-3} \leq \delta^{-3} + (2\sigma - 1 - \delta)^{-3}$$

so that

$$\sum_{\rho \in S_2} |z - \rho|^{-3} \leq \frac{1}{2}(\delta^{-3} + (2\sigma - 1 - \delta)^{-3})|S_2|.$$

Next, each zero in S_3 contributes at most $(2\sigma - 1 - \delta)^{-3}$, and there are $N(1 - \bar{z}, \delta) = N(z, \delta)$ of them, so

$$\sum_{\rho \in S_3} |z - \rho|^{-3} \leq (2\sigma - 1 - \delta)^{-3} N(z, \delta).$$

The main contribution comes from those zeroes in S_4 . We have

$$\sum_{\rho \in S_4} |z - \rho|^{-3} = \int_{\lambda}^{\delta} \frac{dN(z, u)}{u^3} = \frac{N(z, \delta)}{\delta^3} - \frac{N(z, \lambda)}{\lambda^3} + 3 \int_{\lambda}^{\delta} \frac{N(z, u)}{u^4} du.$$

By Lemma 4.5, and since $\delta \leq 1/4$, we have

$$\begin{aligned} 3 \int_{\lambda}^{\delta} \frac{N(z, u)}{u^4} du &\leq 3 \int_{\lambda^*}^{\delta} \frac{6.1u^{3/2} \log T + 0.36 \log \log T - 0.53 \log u + 6}{u^4} du \\ &= 12.2((\lambda^*)^{-3/2} - \delta^{-3/2}) \log T + 0.53 \left(\frac{\log \lambda^*}{(\lambda^*)^3} - \frac{\log \delta}{\delta^3} \right) \\ &\quad + ((\lambda^*)^{-3} - \delta^{-3})(0.36 \log \log T + 5.83). \end{aligned}$$

Collecting these estimates, and recalling that by (3.33) one has

$$|S_2| + 2N(z, \delta) = N(T + 1) - N(T - 1) < 0.71 \log T,$$

we obtain

$$\sum_{\rho: |z-\rho|>\lambda} |z-\rho|^{-3} \leq C_3(\lambda^*, \delta, T) - \frac{N(z, \lambda^*)}{(\lambda^*)^3} \quad (4.29)$$

where

$$\begin{aligned} C_3(\lambda, \delta, T) &:= 12.2(\lambda^{-3/2} - \delta^{-3/2}) \log T + 0.53 \left(\frac{\log \lambda}{\lambda^3} - \frac{\log \delta}{\delta^3} \right) \\ &\quad + (\lambda^{-3} - \delta^{-3})(0.36 \log \log T + 5.83) + (0.355(\delta^{-3} + (2\sigma_3 - 1 - \delta)^{-3}) + 1) \log T. \end{aligned}$$

By inspection $C_3(\lambda, \delta, T)$ is decreasing in λ for fixed δ and $T \geq H_3$. Also, since $t \leq T$, $\lambda^* = \lambda^*(t) \geq \lambda^*(T)$, so that $C_3(\lambda^*, \delta, T) \leq C_3(\lambda^*(T), \delta, T)$. At this point we take $\delta = 1/4$, which is permitted since $\lambda^* < 1/4$. Using a routine computation, we verify that the function

$$h(T) = C_3(\lambda^*(T), 1/4, T) \frac{\log \log T}{(\log T)^2}$$

is decreasing for $T \geq H_3$, so that in particular

$$\sum_{\rho: |z-\rho|>\lambda} |z-\rho|^{-3} \leq h(H_3) \frac{(\log T)^2}{\log \log T} < 28.3 \frac{(\log T)^2}{\log \log T}, \quad (4.30)$$

as required. $\square$

Lemma 4.7. *Suppose Assumption 4.1 holds with respect to a complex zero $\rho_0 = \beta_0 + it$. Let $F_0(z)$ and $F^*(z)$ be as defined in Definition 3.2 and Lemma 3.4 with respect to $\eta = 1 - \beta_0$. If $\lambda^* = \lambda^*(t)$ and $\sigma^* = \sigma^*(t)$ are as defined in (4.21) and (4.22) respectively, and $s_k^* = \sigma^* + ikt$, then*

$$\sum_{1 \leq k \leq K} b_k \left(\sum_{\substack{\rho = \beta + i\gamma \\ \beta \leq \sigma^* - \lambda^*}} \operatorname{Re} F_0(s_k^* - \rho) + \sum_{\substack{\rho = \beta + i\gamma \\ \beta > \sigma^* - \lambda^*}} \operatorname{Re} F^*(s_k^* - \rho) \right) \geq b_1 F^*(\sigma^* - \beta_0) - \frac{0.08}{(\log \log t)^2}.$$

Proof. As before we rewrite the inner sum as

$$\begin{aligned} &\sum_{\substack{\rho = \beta + i\gamma \\ \beta \leq \sigma^* - \lambda^*}} \operatorname{Re} F_0(s_k^* - \rho) + \sum_{\substack{\rho = \beta + i\gamma \\ \beta > \sigma^* - \lambda^*}} \operatorname{Re} F^*(s_k^* - \rho) = \sum_{\rho: |s_k^* - \rho| \leq \lambda^*} \operatorname{Re} F^*(s_k^* - \rho) \\ &\quad + \sum_{\substack{\rho = \beta + i\gamma \\ |s_k^* - \rho| > \lambda^*}} \operatorname{Re} F_0(s_k^* - \rho) + f(0) \sum_{\substack{\rho = \beta + i\gamma \\ \beta \geq \sigma^* - \lambda^* \\ |s_k^* - \rho| > \lambda^*}} \operatorname{Re} \frac{\pi}{2\lambda^*} \cot \left(\frac{\pi(s_k^* - \rho)}{2\lambda^*} \right) \\ &= q_1 + q_2 + q_3, \end{aligned}$$

say.

First consider q_1 . Under hypothesis $\mathcal{H}_1(A)$, and by the definition of σ^* , one has for $1 \leq k \leq K$

$$\zeta(s) \neq 0 \quad (\operatorname{Re} s \geq \sigma^*, |\operatorname{Im} s - kt| \leq T_0). \quad (4.31)$$

Furthermore, for $t \geq H_3$

$$\lambda^*/\eta > \frac{EB^{-2/3}}{A_2 + \varepsilon} \log \log t > 400 = \nu, \quad (4.32)$$

say. Thus, by Lemma 3.16 of the previous chapter,

$$q_1 \geq F^*(s_k^* - \rho) \mathbf{1}_{(k=1)} - C(0, \nu) \eta^3 \frac{N(s_k^*, \lambda^*)}{(\lambda^*)^3}. \quad (4.33)$$

Now suppose ρ is a zero counted by q_2 . By hypothesis $\mathcal{H}_1(A)$, $\sigma^* > 1 - \eta$ and hence $\operatorname{Re}(s_k^* - \rho)/\eta > -1$. Furthermore, by (4.32) one has $|(s_k^* - \rho)/\eta| > \lambda^*/\eta > \nu$. Thus

$$|F_0(s_k^* - \rho)| = |W_0((s_k^* - \rho)/\eta)| \leq \frac{C(-1, \nu) \eta^3}{|s_k^* - \rho|^3},$$

where we recall that W_0 and C are defined in Definition 3.1 and Lemma 2.3 respectively.

Therefore, applying Lemma 4.6,

$$|q_2| \leq C(-1, \nu) \eta^3 \sum_{\substack{\rho = \beta + i\gamma \\ |s_k^* - \rho| > \lambda^*}} |s_k^* - \rho|^{-3} \leq C(-1, \nu) \eta^3 \left(28.3 \frac{(\log kt)^2}{\log \log kt} - \frac{N(s_k^*, \lambda^*)}{(\lambda^*)^3} \right).$$

Since $k \leq K$, $C(-1, \nu) = 50.884 \dots$ and $\eta < (A_2 + \varepsilon)(\log t)^{-2/3}(\log \log t)^{-1/3}$ by Assumption 4.1, one has

$$|q_2| \leq \frac{0.02}{(\log \log t)^2} - C(-1, \nu) \eta^3 \frac{N(s_k^*, \lambda^*)}{(\lambda^*)^3}. \quad (4.34)$$

Lastly, by (4.31) and Lemma 3.15 (of the previous chapter) one has

$$q_3 \geq -10^{-10} \eta > -\frac{10^{-10}}{(\log \log t)^2}. \quad (4.35)$$

Combining (4.33), (4.34) and (4.35) and summing over k , one obtains

$$\sum_{1 \leq k \leq K} b_k \left(\sum_{\substack{\rho = \beta + i\gamma \\ \beta \leq \sigma^* - \lambda^*}} \operatorname{Re} F_0(s_k^* - \rho) + \sum_{\substack{\rho = \beta + i\gamma \\ \beta > \sigma^* - \lambda^*}} \operatorname{Re} F^*(s_k^* - \rho) \right) > b_1 F^*(\sigma^* - \beta_0) - \frac{0.08}{(\log \log t)^2},$$

so the result follows. $\square$

Recall that for any fixed σ and $\lambda \in (1 - \sigma, 1/2)$, one has (3.8), i.e.

$$0 \leq \sum_{1 \leq k \leq K} b_k I(s_k, \lambda) + a_0 Q(\sigma).$$

At this point we take $\sigma = \sigma^*$ and $\lambda = \lambda^*$. Applying Lemma 4.4, Lemma 4.7 and Lemma 3.18 with this choice of parameters, for all $t \geq H_3$ one has

$$\begin{aligned} & \sum_{1 \leq k \leq K} b_k I(s_k^*, \lambda^*) \\ & \leq \frac{f(0)}{4\lambda^*} \sum_{1 \leq k \leq K} b_k \int_{-\infty}^{\infty} \frac{\log |\zeta(s_k^* - \lambda^* + 2\lambda^* iu/\pi)| - \log |\zeta(s_k^* + \lambda^* + 2\lambda^* iu/\pi)|}{(\cosh u)^2} du \\ & \quad - \sum_{1 \leq k \leq K} b_k \left(\sum_{\substack{\rho = \beta + i\gamma \\ \beta > \sigma^* - \lambda^*}} \operatorname{Re} F^*(s_k^* - \rho) + \sum_{\substack{\rho = \beta + i\gamma \\ \beta \leq \sigma^* - \lambda^*}} \operatorname{Re} F_0(s_k^* - \rho) \right) \\ & \quad + \operatorname{Re} \sum_{1 \leq k \leq K} b_k F^*(s_k^* - 1) + \sum_{1 \leq k \leq K} b_k |E(s_k^*)| \\ & \leq 9.108 f(0) (\log t)^{2/3} (\log \log t)^{1/3} + 1.263 f(0) \frac{(\log t)^{2/3}}{(\log \log t)^{2/3}} - b_1 F^*(\sigma^* - \beta_0) \\ & \quad + \frac{0.08}{(\log \log t)^2} + \sum_{1 \leq k \leq K} b_k |E(s_k^*)|. \end{aligned}$$

By Lemma 2.7, one has

$$\sum_{1 \leq k \leq K} b_k |E(s_k)| \leq (14\eta^2 + 425\eta^3) \sum_{1 \leq k \leq K} b_k < 55\eta^2,$$

and by Lemma 3.20, one has

$$Q(\sigma^*) \leq F(\sigma^* - 1) - 2.9\eta.$$

Summing these estimates, one obtains

$$0 \leq 9.108 f(0) (\log t)^{2/3} (\log \log t)^{1/3} + a_0 F(\sigma^* - 1) - a_1 F^*(\sigma^* - \beta_0) + C_3(\eta, t) \quad (4.36)$$

where

$$C_3(\eta, t) := 1.263 \eta w(0) \frac{(\log t)^{2/3}}{(\log \log t)^{2/3}} + \frac{0.08}{(\log \log t)^2} + 55\eta^2 - 2.9a_0\eta.$$

For each fixed $t \geq H_3$, C_3 is increasing in η . By Assumption 4.1 one has $\eta \leq (A_2 + \varepsilon)(\log t)^{-2/3}(\log \log t)^{-1/3}$; substituting this bound gives (for any $x \geq \log H_3$)

$$C_3(\eta, e^x) \leq \frac{0.139}{\log x} + \frac{0.08}{(\log x)^2} - \frac{0.05}{x^{2/3}(\log x)^{1/3}} + \frac{0.03}{x^{4/3}(\log x)^{2/3}}.$$

However the function on the right side is decreasing for $x \geq \log H_3$, so

$$C_3(\eta, t) \leq C_3(\eta, H_3) = 0.0109804538 \dots \quad (t \geq H_3).$$

Next, to estimate $F^*(\sigma^* - \beta_0)$, we use Assumption 4.1, the definition of σ^* , $A \leq A_2$ and $t \geq H_3$ to get

$$\frac{\pi(\sigma^* - \beta_0)}{2\lambda^*} < \frac{\pi}{2\lambda^*} \left(\frac{A + \varepsilon}{(\log t)^{2/3}(\log \log t)^{1/3}} - \frac{A}{L_1^{2/3}L_1^{1/3}} \right) < 10^{-7}.$$

Since $1/x - \cot x < 10^{-7}$ for $0 < x < 10^{-7}$, we have

$$F^*(\sigma^* - \beta_0) \geq F(\sigma^* - \beta_0) - \frac{10^{-7}w(0)\eta}{\lambda^*} > F(\sigma^* - \beta_0) - 10^{-8}. \quad (4.37)$$

Lastly, we estimate the quantity $a_1F(\sigma^* - \beta_0) - a_0F(\sigma^* - 1)$ by repeating the analysis of the previous chapter (with our new value of θ and using $\mu := (1 - \sigma^*)/\eta \geq 0.9999953403 \dots$) to obtain

$$a_1F(\sigma^* - \beta_0) - a_0F(\sigma^* - 1) = \int_0^{2\theta \cot \theta} e^{\mu u} (a_1 e^{-u} - a_0) w(u) du > 1.01267.$$

Combining the above estimate with (4.36) and (4.37), one finally has

$$\begin{aligned} (1 - \beta_0)(\log t)^{2/3}(\log \log t)^{1/3} &> \frac{1.01267 - 10^{-8}a_1 - C_3(\eta, t)}{9.108w(0)} \\ &> 0.01945 > A_2 + \varepsilon \geq A + \varepsilon, \end{aligned}$$

so we have reached the desired contradiction to Assumption 4.1. Thus, Lemma 4.1 (and Theorem 1.3) is proved.

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