Every admissible two-row half-shifted cup coordinate is coefficientwise nonnegative

Kenan

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Abstract

For the flagged array $A_{r,j}(a) = (r+1)h_{2j-r-1}(a, a+1, \ldots, a+r+1)$, every admissible two-row coordinate of the gauged inverse noncrossing-cup transform is a polynomial with nonnegative rational coefficients in $b = a - 1/2$. The proof converts exact cup incidences into uniform recurrences, controls their alternating corrections by coefficientwise-positive strip and arm gaps, and cancels the remaining denominator inside the principal generalized-Vandermonde product.

$$\alpha_{(n,k)} \in \mathbf{Q}_{\geq 0}\left[a - \tfrac{1}{2}\right] \quad (1 \leq k \leq n,\ d \geq \max(n, k+1))$$

1 The two-row statement

Fix a dimension d and the flagged array A. For a partition λ, the minor $H_\lambda(a)$ is selected by the row set K_λ, and the vector of signed maximal minors is transformed through the inverse gauged cup-incidence matrix. The resulting coordinate attached to the canonical matching for λ is denoted $\alpha_\lambda(a)$.

The theorem treats every two-row partition (n, k) allowed by $1 \leq k \leq n$ and $d \geq \max(n, k+1)$. After the half-shift $b = a - 1/2$, its coordinate has nonnegative rational coefficients and is nonzero, hence it is strictly positive whenever $a \geq 1/2$.

$$A_{r,j}(a) = (r+1)h_{2j-r-1}(a, a+1, \ldots, a+r+1)$$

$$\alpha_{(n,k)} \in \mathbf{Q}_{\geq 0}[b], \qquad b = a - \frac{1}{2}$$

2 Cup rows become recurrences

The selected and unselected positions for (n, k) form a stable word whose compatible noncrossing matchings can be enumerated recursively. Removing the forced outer pair leaves four interior incidences, three incidences on the diagonal, and five on the subdiagonal, with exact gauged signs.

Writing $S_{n,k} = \sum_{j=0}^{k}(-1)^j H_{(n,j)}$, those cup identities collapse to three coordinate recurrences. They cover the interior, diagonal, and subdiagonal and reduce the entire two-row sector to one alternating sum of neighboring minors.

$$\alpha_{(n,k)} = \alpha_{(n-1,k)} + (-1)^n S_{n,k} \qquad (n \geq k+2)$$

$$\alpha_{(k,k)} = \alpha_{(k-1,k-1)} + (-1)^k S_{k,k}$$

$$\alpha_{(k+1,k)} = \alpha_{(k-1,k-1)} + \alpha_{(k,k)} + (-1)^{k+1} S_{k+1,k}$$

3 A uniform neighboring minor

Arithmetic divided differences reduce the neighboring two-row minor to a two-by-two Newton-tail determinant. Relative to the principal minor $H_\varnothing = P_d(b)$, every pair (n,k) has the same denominator pattern $\Delta_{n,k}$ and an explicit positive amplitude $Z_{n,k}$.

All factors in the displayed ratio for $Z_{n,k}/Z_{n,0}$ are positive on the admissible domain. This supplies the exact quantitative input used by the correction recurrence rather than appealing to generic Temperley–Lieb or flagged Jacobi–Trudi positivity.

$$P_d(b) = d! \prod_{0 \leq p < q \leq d} (2b + p + q + 1)$$

$$\frac{H_{(n,k)}}{H_\varnothing} = \frac{Z_{n,k}}{\Delta_{n,k}}, \qquad \Delta_{n,k} = (Y-1)(Y)_{k-1}(Y)_n, \qquad Y = 2b + d + 1$$

$$\frac{Z_{n,k}}{Z_{n,0}} = \frac{(d+k-1)(n-k+1)(d^2+d-k(n+1))}{k\, d(d-1)(n+1)} \binom{d}{k-1}$$

4 Positive corrections and arm gaps

Clearing the common denominator in the alternating minor sum defines $C_{n,k}$. Consecutive denominators cancel to give a first-order recurrence with alternating $Z_{n,k}$ terms. Pairing odd steps reduces positivity to the strip gap $L_{n,k}$, which is coefficientwise nonnegative because the exact amplitude ratio is bounded by the adjacent linear factor.

To move along the first row length n, define the arm gap $E_{n,k}$. Its recurrence has the same alternating shape. The paired odd remainder has explicitly nonnegative quadratic, linear, and constant coefficients, so both $C_{n,k}$ and $E_{n,k}$ lie in $\mathbf{Q}_{\geq 0}[b]$ throughout the admissible domain.

$$C_{n,k} = (Y+k-2)C_{n,k-1} + (-1)^k Z_{n,k}$$

$$L_{n,k} = (Y + k - 2)Z_{n,k-1} - Z_{n,k} \in \mathbf{Q}_{\geq 0}[b]$$

$$E_{n,k} = (Y + n - 1)C_{n-1,k} - C_{n,k} \in \mathbf{Q}_{\geq 0}[b]$$

1. Pair each odd correction step with the preceding even step.
2. Use the exact strip-ratio inequality to make the paired correction coefficientwise nonnegative.
3. Use the arm-ratio factorization and the boundary inequality for constant terms to make the paired arm remainder coefficientwise nonnegative.

5 Closing every coordinate

Normalize the coordinate by $T_{n,k} = \Delta_{n,k}\alpha_{(n,k)}/H_\varnothing$. The diagonal recurrence starts from the positive one-column base, the subdiagonal recurrence is controlled by $E_{k+1,k}$, and each interior odd step pairs with its preceding even step to leave $E_{n,k}$. Thus every normalized coordinate is coefficientwise nonnegative and nonzero.

It remains to remove the normalization. Distinct factors of the principal product P_d realize every factor of $\Delta_{n,k}$ when $d \geq \max(n, k+1)$. Therefore $P_d/\Delta_{n,k}$ is itself a product of positive linear polynomials in b, and multiplying it by $T_{n,k}$ proves the theorem.

$$T_{n,k} = \frac{\Delta_{n,k}\alpha_{(n,k)}}{H_\varnothing}$$

$$\Delta_{n,k} \mid P_d(b)$$

$$\alpha_{(n,k)} = \frac{P_d(b)}{\Delta_{n,k}} T_{n,k} \in \mathbf{Q}_{\geq 0}[b]$$

6 What the certificate checks

The pinned certificate checks direct maximal minors, independently reconstructed cup coordinates, Newton-tail identities, full cup incidence, matching stability, amplitude formulas, and direct coordinate polynomials within its immutable finite bounds. It also checks the exact denominator and positivity identities used by the uniform proof.

A counterfeit preserves pointwise positivity and denominator divisibility while introducing a negative shifted coefficient. Its rejection records why those weaker properties cannot replace the coefficientwise recurrence argument. The literature search separately distinguishes this arithmetic half-shifted inverse-cup theorem from classical generalized-Vandermonde, divided-difference, Temperley–Lieb immanant, and inverse-incidence machinery.

$$d \le 7 \text{ for direct minors and full cup incidence,} \qquad d \le 32 \text{ for matching stability}$$

$$\text{pointwise positivity} + \text{denominator divisibility} \not\Rightarrow \mathbf{Q}_{\ge 0}[b]$$

7 Pinned certificate

The pinned verifier cross-checks the determinant, cup-incidence, recurrence, amplitude, denominator-cancellation, and shifted-coefficient seams at fixed finite bounds, with pinned parent verifiers and transcript digests. A denominator-divisible, pointwise-positive counterfeit is required to fail the coefficient checker.

Run command

```
nice -n 19 timeout 360 uv run --frozen python canon/witnesses/C-0112/verify.py --direct-max-dim
```

Pinned files

- `canon/witnesses/C-0112/verify.py`
- `canon/witnesses/C-0112/results.json`
- `canon/witnesses/C-0112/PIN.md`
- `canon/witnesses/C-0110/verify.py`
- `canon/witnesses/C-0071/verify.py`
- `canon/claims/C-0112-half-shifted-all-two-row-cup-positivity.md`

8 Scope

Coefficientwise half-shifted positivity is established for every admissible two-row partition in the flagged array.

9 Sources

- Canonical claim: `canon/claims/C-0112-half-shifted-all-two-row-cup-positivity.md`
- Result: `scratch/adjacent-unconditional--half-shifted-all-two-row-cup-positivity/RESULT.md`
- Dated literature search: `literature/2026-07-19-half-shifted-all-two-row-cup-positivity-search.md`

- Source and verifier pin: `canon/witnesses/C-0112/PIN.md`