Third-strip two-row half-shifted cup positivity in every area

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Abstract

For every $n \geq 3$ and $d \geq \max(n, 4)$, the gauged noncrossing-cup coordinate indexed by the two-row partition $(n, 3)$ is a polynomial with nonnegative rational coefficients in $b = a - \frac{1}{2}$. The proof derives a stable four-state cup recurrence, evaluates the third-strip neighboring minor, reduces positivity to a cleared recurrence over the previously proved $(n, 2)$ family, and cancels every denominator factor against the principal product.

$$\alpha_{(n,3)}(a) \in \mathbf{Q}_{\geq 0}[a - \tfrac{1}{2}]$$

1 The third-strip statement

The flagged array, maximal minors, and gauged cup coordinates are the same objects used in the preceding strip results. The target here is the coordinate indexed by $(n, 3)$ throughout every allowed area d.

The conclusion is coefficientwise: after shifting to $b = a - \frac{1}{2}$, the exact coordinate belongs to $\mathbf{Q}_{\geq 0}[b]$. It is consequently strictly positive for every $a \geq \frac{1}{2}$.

$$A_{k,j}(a) = (k+1)h_{2j-k-1}(a, a+1, \ldots, a+k+1)$$

$$\alpha_{(n,3)} \in \mathbf{Q}_{\geq 0}[b], \qquad b = a - \frac{1}{2}$$

2 Find the stable cup recurrence

For $n \geq 5$, balance around the two isolated selected endpoints leaves exactly four compatible states: $(n-1, 2)$, $(n-1, 3)$, $(n, 2)$, and $(n, 3)$. Their gauged signs alternate.

Combining that incidence row with the C-0106 recurrence for the second strip closes the new coordinate on its predecessor and the alternating sum of four neighboring minors. The small arms $n = 3$ and $n = 4$ have separate stable rows because additional balanced matchings occur there.

$$\alpha_{(n-1,2)} - \alpha_{(n-1,3)} - \alpha_{(n,2)} + \alpha_{(n,3)} = (-1)^{n+3} H_{(n,3)}$$

$$\alpha_{(n,3)} = \alpha_{(n-1,3)} + (-1)^n \big(H_{(n)} - H_{(n,1)} + H_{(n,2)} - H_{(n,3)}\big)$$

3 Evaluate the third-strip minor

Removing the common principal block leaves a two-by-two Newton-tail determinant with target orders $d+2$ and $d+n$. The divided-difference identity converts it to an explicit rational expression.

The numerator amplitude $Z_n(d)$ factors into nonnegative terms on $n \geq 3$ and $d \geq \max(n, 4)$. Its denominator adds the extra factor $Y + 1$ expected one strip farther into the two-row family.

$$\frac{H_{(n,3)}}{H_\varnothing} = \frac{Z_n(d)}{Y(Y+1)(Y-1)_{n+1}}$$

$$Z_n(d) = \frac{(d+2)(d+n)(n-2)(d^2+d-3n-3)\prod_{r=1}^{n-1}(d-r)}{6(n+1)!}$$

4 Clear the recurrence

Set $m = d - n$ and clear the complete third-strip denominator. The resulting polynomial $T_n(m, b)$ satisfies a recurrence whose correction is built from the positive second-strip correction K_n and the new amplitude Z_n.

Although that definition contains a subtraction, exact simplification factors it as $G_n(m)J_n(m, b)$. Every coefficient of the displayed cubic J_n is positive on the theorem's domain, so the correction L_n is coefficientwise nonnegative.

$$T_n(m, b) = \frac{Y(Y+1)(Y-1)_{n+1}\alpha_{(n,3)}}{H_\varnothing}$$

$$T_n = (2b + m + 2n)T_{n-1}(m + 1, b) + (-1)^n L_n(m, b)$$

$$L_n(m, b) = G_n(m)J_n(m, b) \in \mathbf{Q}_{\geq 0}[b, m]$$

5 Handle the exceptional bases and odd steps

Direct substitution of the stable $n = 3$ row gives an explicitly coefficientwise-positive polynomial T_3. The exceptional $n = 4$ row contributes an additional correction E_4, displayed in the canonical proof as a product of coefficientwise-positive factors.

For odd $n \geq 5$, pairing two recurrence steps produces a remainder R_n. After extracting a positive rising-factorial prefactor, each coefficient of the remaining polynomial becomes positive after writing $n = q + 3$. Even steps are positive directly, so these bases and paired odd steps complete the induction.

$$T_4 = (2b + m + 8)T_3(m + 1, b) + L_4(m, b) + E_4(m, b)$$

$$R_n = (2b + m + 2n)L_{n-1}(m + 1, b) - L_n(m, b) \in \mathbf{Q}_{\geq 0}[b, m]$$

$$T_n \in \mathbf{Q}_{\geq 0}[b, m] \qquad (n \geq 3)$$

6 Cancel the denominator and return to the coordinate

The cleared denominator contains single factors at $2b + d$ and from $2b + d + 3$ onward, together with double factors at $2b + d + 1$ and $2b + d + 2$. The canonical proof assigns distinct pairs in the generalized-Vandermonde principal product to every one of these factors.

The denominator therefore divides the principal product, and the quotient has strictly positive coefficients. Multiplying by the nonnegative cleared polynomial proves the original third-strip coordinate is coefficientwise nonnegative and has a positive constant term.

$$P_d(b) = d! \prod_{0 \leq p < q \leq d} (2b + p + q + 1)$$

$$\alpha_{(n,3)} \in \mathbf{Q}_{\geq 0}[b] \quad (n \geq 3,\ d \geq \max(n, 4))$$

7 Pinned certificate

The pinned certificate checks direct maximal minors, independent low-rank coordinates, Newton-tail identities, full cup incidence, matching stability, the symbolic recurrence and odd-step remainder, denominator divisibility, and direct coordinate polynomials. Its negative controls reject altered determinants, missing cup matchings, perturbed corrections, weakened bounds, and a disabled coefficient checker.

Run command

```
nice -n 19 timeout 360 uv run --frozen python canon/witnesses/C-0110/verify.py --direct-max-di
```

Pinned files

- `canon/witnesses/C-0110/verify.py`
- `canon/witnesses/C-0110/PIN.md`
- `canon/witnesses/C-0110/results.json`
- `canon/witnesses/C-0106/verify.py`
- `canon/witnesses/C-0071/verify.py`
- `canon/claims/C-0110-half-shifted-two-row-third-strip-cup-positivity-all-areas.md`

8 Scope

Coefficientwise half-shifted positivity is established for the two-row family $(n, 3)$ whenever $n \geq 3$ and $d \geq \max(n, 4)$.

9 Sources

- Canonical claim: `canon/claims/C-0110-half-shifted-two-row-third-strip-cup-positivity-all-areas` `md`
- Certificate pin: `canon/witnesses/C-0110/PIN.md`
- Recorded search: `literature/2026-07-19-half-shifted-two-row-third-strip-cup-all-area-search.` `md`
- Proof receipt: `scratch/adjacent-unconditional--half-shifted-two-row-third-strip-cup-all-area-p` `RESULT.md`