An exact phase diagram for the JustZeta comparisons

Kenan

2026-07-19

Abstract

Garcia–Grenie–Molteni proved explicit rational lower and upper envelopes $L(s) < (s-1)\zeta(s) < U(s)$ for $s > 1$. This result reconstructs that source envelope from pinned analytic inputs, proves that $U(s)$ is globally below $e^{\gamma(s-1)}$, and gives the exact two-root sign diagram for the comparison of $L(s)$ with $(s+1)/2$.

$$U(s) < e^{\gamma(s-1)}\ (s > 1), \qquad L(s) > \frac{s+1}{2} \iff s \in (1, s_-) \cup (s_+, \infty)$$

1 The source envelope and the comparison problem

Let $F(s) = (s-1)\zeta(s)$ for $s > 1$. The source theorem places F strictly between two displayed rational functions L and U.

The new task is not to replace that analytic envelope. It is to classify exactly where its upper and lower functions improve two classical comparators.

$$L(s) = s - (1-\gamma)\frac{s-1}{s} - \frac{(s-1)^2}{s^2}$$

$$U(s) = s - (1-\gamma)\frac{s-1}{s} - \frac{(s-1)^2}{3s^2}$$

$$L(s) < F(s) < U(s) \qquad (s > 1)$$

2 Reconstruct the envelope without decimal crossover labels

On $1 < s \le 3$, write $t = s-1$ and use the pinned five-term Stieltjes expansion together with the displayed Zhang–Williams remainder bound. Multiplying each comparison by a positive denominator reduces it to a degree-six polynomial.

All seven Bernstein coefficients are rigorously positive for both inequalities. For larger s, the source's fractional-part integral identities give elementary upper and lower baselines; exact rational Arb cells bridge finite intervals and monotone estimates close the tails.

$$F(1+t) = 1 + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_n}{n!} t^{n+1}$$

$$|F(1+t) - P_5(t)| \le \frac{4\sqrt{2}\, c^5 t^6}{5(1-ct)} \qquad (0 \le t \le 2)$$

$$s - 1 + \frac{s+3}{2^{s+1}} < F(s) < s - 1 + \frac{s+1}{2^s} \qquad (s > 3)$$

3 Turn the upper comparison into one positive quadratic

For $t = s - 1 > 0$, the quadratic Taylor polynomial is a strict lower bound for $e^{\gamma t}$. Subtracting $U(1+t)$ factors the difference into positive elementary terms and a quadratic $Q(1+t)$.

The leading coefficient of Q is positive and its discriminant is negative. Therefore Q is positive on the whole real line, making the upper comparison global rather than restricted to the finite interval printed in the source.

$$e^{\gamma t} > 1 + \gamma t + \frac{\gamma^2 t^2}{2}$$

$$1 + \gamma t + \frac{\gamma^2 t^2}{2} - U(1+t) = \frac{t^2}{6(1+t)^2} Q(1+t)$$

$$Q(x) = 3\gamma^2 x^2 - 6(1-\gamma)x + 2, \qquad \mathrm{disc}(Q) = 12(\gamma^2 - 6\gamma + 3) < 0$$

4 Factor the lower comparison exactly

The difference between $L(1+t)$ and the classical lower bound has a positive prefactor times one quadratic. Its discriminant is positive, and both roots lie in $t > 0$.

Since the quadratic opens upward, its sign is positive before the first root and after the second, zero at the roots, and negative between them. Translating back to s yields the complete phase diagram, including the second dominance interval beyond s_+.

$$L(1+t) - \frac{2+t}{2} = \frac{t}{2(1+t)^2} \left(t^2 + (2\gamma - 2)t + (2\gamma - 1)\right)$$

$$s_\pm = 2 - \gamma \pm \sqrt{\gamma^2 - 4\gamma + 2}$$

$$L(s) > \frac{s+1}{2} \iff 1 < s < s_- \text{ or } s > s_+$$

5 Locate the two transitions

Directed enclosures isolate each algebraic transition value. Equality occurs only at those two values, and the lower comparison reverses on the interval between them.

These enclosures document the location of the exact roots; the proof of the sign diagram itself comes from the symbolic factorization and root ordering, not from sampling decimal points.

$$1.2668508101540 < s_- < 1.2668508101541$$

$$1.5787178600428 < s_+ < 1.5787178600429$$

6 Role of the certificate

The main witness authenticates the frozen preprint bytes, reconstructs the strict source envelope, checks every Bernstein coefficient, finite large-s cell, tail inequality, discriminant, root enclosure, and sign interval, and writes the standard eight-check acceptance record.

A dedicated counterfeit rejects an inward root enclosure and an endpoint-positive but interior-negative polynomial. A separate PARI/GP reconstruction corroborates the Stieltjes data, exact algebra, finite cells, tails, and roots without importing the Arb verifier.

7 Pinned certificate

The pinned witness verifies the source envelope from displayed analytic inputs rather than source computation labels, then certifies the global exponential comparison and exact two-root lower phase diagram. Independent counterfeit and PARI/GP paths test the two principal false shortcuts and reconstruct the route separately.

Run command

```
uv run --frozen python canon/witnesses/C-0109/verify.py --bits 384 --finite-cells 512 --results
```

Pinned files

- `canon/witnesses/C-0109/verify.py`
- `canon/witnesses/C-0109/PROOF.md`

- canon/witnesses/C-0109/PIN.md
- canon/witnesses/C-0109/theorem.json
- canon/witnesses/C-0109/counterfeit.py
- canon/witnesses/C-0109/pari_reconstruction.py
- canon/witnesses/C-0109/source/arxiv-2607.08342v1.pdf
- canon/claims/C-0109-garcia-grenie-molteni-just-zeta-complete-dominance.md

8 Scope

The result gives the exact comparison ranges obtained from the fixed Garcia–Grenie–Molteni lower and upper envelopes for real $s > 1$.

9 Sources

- Canonical claim: canon/claims/C-0109-garcia-grenie-molteni-just-zeta-complete-dominance.md
- Proof: canon/witnesses/C-0109/PROOF.md
- Certificate pin: canon/witnesses/C-0109/PIN.md
- Pinned source: canon/witnesses/C-0109/source/arxiv-2607.08342v1.pdf
- Literature search: literature/2026-07-19-garcia-grenie-molteni-just-zeta-complete-dominance-sea md