Second-strip two-row half-shifted cup positivity in every area

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Abstract

For every $n \geq 2$ and $d \geq \max(n, 3)$, the gauged noncrossing-cup coordinate indexed by the two-row partition $(n, 2)$ is a polynomial with nonnegative rational coefficients in $b = a - \frac{1}{2}$. The proof converts the cup-coordinate problem into a four-incidence recurrence, evaluates the neighboring flagged minor by a two-by-two Newton-tail determinant, proves positivity of the cleared recurrence, and cancels its denominator against the principal product.

$$\alpha_{(n,2)}(a) \in \mathbf{Q}_{\geq 0}[a - \tfrac{1}{2}]$$

1 The coordinate and the claim

Start with the flagged array $A_{k,j}(a) = (k+1)h_{2j-k-1}(a, a+1, \ldots, a+k+1)$. Its maximal minors $H_\lambda(a)$ are indexed by partitions, and inverse noncrossing incidence expresses them in the gauged cup coordinates $\alpha_\lambda(a)$.

For the second-strip family, the assertion is stronger than pointwise positivity: after the half-shift $b = a - \frac{1}{2}$, every coefficient of the exact polynomial is nonnegative. This immediately gives strict positivity for $a \geq \frac{1}{2}$.

$$A_{k,j}(a) = (k+1)h_{2j-k-1}(a, a+1, \ldots, a+k+1)$$

$$\alpha_{(n,2)} \in \mathbf{Q}_{\geq 0}[b], \qquad b = a - \frac{1}{2}$$

2 Reduce the cup row to four neighboring states

For $n \geq 4$, balance around the two isolated selected endpoints forces each endpoint to use one of its two adjacent unselected neighbors. After those choices are deleted, the remaining matching is the unique rainbow matching.

The resulting four cup states are $(n-1, 1)$, $(n, 1)$, $(n-1, 2)$, and $(n, 2)$. Combining their signed incidence row with the previously established hook recurrence closes the second-strip coordinate on the preceding second-strip coordinate and three neighboring minors.

$$\alpha_{(n-1,1)} - \alpha_{(n,1)} - \alpha_{(n-1,2)} + \alpha_{(n,2)} = (-1)^n H_{(n,2)}$$

$$\alpha_{(n,2)} = \alpha_{(n-1,2)} + (-1)^n \big(H_{(n)} - H_{(n,1)} + H_{(n,2)}\big)$$

3 Evaluate the neighboring minor

Removing the common principal block from the row set leaves a two-by-two Newton-tail determinant with target orders $d+1$ and $d+n$. Arithmetic divided differences turn every entry into a binomial coefficient divided by a rising factorial.

The determinant simplifies to an explicit amplitude $W_n(d)$ over the denominator $Y(Y-1)_{n+1}$, where $Y = 2b + d + 1$. On the theorem's domain, the amplitude factors appearing in the canonical formula have the required signs.

$$\frac{H_{(n,2)}}{H_\varnothing} = \frac{W_n(d)}{Y(Y-1)_{n+1}}$$

$$W_n(d) = \frac{(d+1)(d+n)(d^2+d-2n-2)\prod_{r=2}^{n-1}(d-r)}{2n(n+1)(n-2)!}$$

4 Clear denominators and prove the recurrence positive

With $m = d - n$, multiply the coordinate by the complete Newton denominator and divide by the principal minor. The resulting polynomial $S_n(m,b)$ satisfies a one-step recurrence whose correction $K_n(m,b)$ has an explicit quadratic expansion in b.

Every factor and coefficient in that expansion is nonnegative for $m \geq 0$. Even recurrence steps are therefore immediately positive. The canonical proof supplies a coefficientwise-positive base at $n = 3$.

$$S_n(m,b) = \frac{Y(Y-1)_{n+1}\alpha_{(n,2)}}{H_\varnothing}$$

$$S_n = (2b + m + 2n)S_{n-1}(m+1,b) + (-1)^n K_n(m,b)$$

$$K_n(m,b) \in \mathbf{Q}_{\geq 0}[b,m]$$

5 Pair the odd steps

At an odd index the correction enters with a minus sign, so one-step positivity alone is insufficient. Pairing an odd step with the preceding even step leaves a remainder R_n.

The proof rewrites this remainder as the nonnegative hook remainder from C-0104 plus an amplitude difference. After substituting $d = m+n$, the remaining numerator is a polynomial whose coefficients are positive for every odd $n \geq 5$. This completes the induction without replacing coefficientwise positivity by sampled or pointwise checks.

$$S_n = (2b + m + 2n)(2b + m + 2n - 1)S_{n-2}(m + 2, b) + R_n(m, b)$$

$$R_n(m, b) \in \mathbf{Q}_{\geq 0}[b, m]$$

6 Cancel the principal factors

The denominator used to define S_n is not left as a rational-function artifact. Distinct pairs in the generalized-Vandermonde principal product supply every linear factor, including the repeated factor at $2b + d + 1$.

After cancellation, the remaining principal quotient has strictly positive coefficients. Multiplying it by the nonnegative cleared coordinate proves the original cup coordinate lies in $\mathbf{Q}_{\geq 0}[b]$ and is positive at and beyond the half-shift boundary.

$$P_d(b) = d! \prod_{0 \leq p < q \leq d} (2b + p + q + 1)$$

$$\alpha_{(n,2)} \in \mathbf{Q}_{\geq 0}[b] \quad (n \geq 2,\ d \geq \max(n, 3))$$

7 Pinned certificate

The pinned certificate checks direct flagged minors, the Newton-tail identity, complete cup incidence in bounded ranks, matching stability, the symbolic strip recurrence, denominator divisibility, and direct coordinate polynomials. It also rejects determinant, cup-support, recurrence, and coefficient-checker mutations; no finite collection of positive evaluations is accepted as a substitute for the exact coefficientwise argument.

Run command

```
uv run --frozen python canon/witnesses/C-0106/verify.py
```

Pinned files

- `canon/witnesses/C-0106/verify.py`
- `canon/witnesses/C-0106/PIN.md`

- canon/witnesses/C-0106/results.json
- canon/witnesses/C-0104/verify.py
- canon/claims/C-0106-half-shifted-two-row-second-strip-cup-positivity-all-areas.md

8 Scope

Coefficientwise half-shifted positivity is established for the two-row family $(n, 2)$ whenever $n \geq 2$ and $d \geq \max(n, 3)$.

9 Sources

- Canonical claim: canon/claims/C-0106-half-shifted-two-row-second-strip-cup-positivity-all-area md
- Certificate pin: canon/witnesses/C-0106/PIN.md
- Recorded search: literature/2026-07-19-half-shifted-two-row-second-strip-cup-all-area-search. md
- Proof receipt: scratch/adjacent-unconditional--half-shifted-two-row-second-strip-cup-all-area- RESULT.md