Two-row-hook half-shifted cup positivity in every area

Kenan

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Abstract

For every $d \geq n \geq 2$, the gauged noncrossing-cup coordinate indexed by the two-row hook $(n, 1)$ is coefficientwise nonnegative in $b = a - 1/2$ and is positive even at $b = 0$. The proof identifies an exact four-incidence cup row, evaluates a two-by-two Newton-tail determinant, derives a cleared hook recurrence, pairs odd and even steps to expose a positive remainder, and cancels the denominator against the principal minor.

$$\alpha_{(n,1)} \in \mathbf{Q}_{\geq 0}[a - \tfrac{1}{2}] \quad \text{and} \quad \alpha_{(n,1)} > 0 \quad (d \geq n \geq 2,\ a \geq \tfrac{1}{2})$$

1 The first interior all-area family

The coordinate $\alpha_{(n,1)}$ belongs to the same flagged-minor and gauged noncrossing-cup transform as the row and column families. Unlike those boundary partitions, $(n, 1)$ is an interior two-row hook.

The theorem states coefficientwise positivity in $b = a - 1/2$ for every admissible dimension and hook length, together with strict positivity on the full half-shifted boundary $b \geq 0$.

$$\alpha_{(n,1)} \in \mathbf{Q}_{\geq 0}[b], \qquad d \geq n \geq 2$$

$$\alpha_{(n,1)}(b) > 0 \qquad (b \geq 0)$$

2 Four compatible cup incidences

For $n \geq 3$, the lifted word has two isolated selected endpoints. Noncrossing balance forces each isolated endpoint to pair with its immediate left or right neighbor.

The four left/right choices produce exactly the matchings indexed by $(n - 1)$, (n), $(n - 1, 1)$, and $(n, 1)$. Their gauged signs are $+1, -1, -1, +1$, giving a four-term incidence relation.

$$\alpha_{(n-1)} - \alpha_{(n)} - \alpha_{(n-1,1)} + \alpha_{(n,1)} = (-1)^{n+1} H_{(n,1)}$$

$$\alpha_{(n,1)} = \alpha_{(n-1,1)} + (-1)^n \big(H_{(n)} - H_{(n,1)}\big)$$

3 The exceptional base and neighboring minor

At $n = 2$, an extra balanced matching appears, so the general four-incidence argument is replaced by the already established area-three base formula.

For $n \geq 3$, a Newton-column change reduces $H_{(n,1)}$ to a two-by-two tail determinant with target orders d and $d + n$. Evaluating it gives one closed rational ratio against the principal minor.

$$\alpha_{(2,1)} = 2H_\varnothing - H_{(1)} + H_{(2)} + H_{(1,1)} - H_{(2,1)}$$

$$\frac{H_{(n,1)}}{H_\varnothing} = \frac{V_n(d)}{(Y-1)_{n+1}}$$

$$V_n(d) = \frac{(d+n)(d^2 + d - n - 1)\prod_{r=2}^{n-1}(d-r)}{(n+1)(n-1)!}$$

4 A cleared hook recurrence

Set $m = d - n$ and clear the common denominator by defining $T_n = (Y-1)_{n+1}\alpha_{(n,1)}/H_\varnothing$. Combining the hook cup relation with the one-row formula from C-0102 produces a first-order recurrence.

Its correction term contains the exact cancellation between the row and hook minor amplitudes. The base T_2 is an explicit polynomial with positive coefficients.

$$T_n(m,b) = (2b + m + 2n)T_{n-1}(m+1,b) + (-1)^n C_n(m,b)$$

$$C_n(m,b) = \frac{(m+2n)(m+1)_{n-1}}{n!}\left(2b + \frac{m(m-1)}{(n+1)(m+n-1)}\right)$$

5 Pairing parity exposes positivity

When n is even, the one-step recurrence is coefficientwise nonnegative for every integer $m \geq 0$. When n is odd, substituting the preceding even step converts the subtraction into a two-step recurrence.

The resulting remainder $E_n(m,b)$ has explicitly nonnegative coefficients of b^2, b, and 1, each factored into nonnegative expressions in m and n. Induction from T_2 therefore proves coefficientwise positivity and strict pointwise positivity.

$$T_n = (2b + m + 2n)(2b + m + 2n - 1)T_{n-2}(m + 2, b) + E_n(m, b)$$

$$E_n(m, b) \in \mathbf{Q}_{\geq 0}[b, m]$$

6 Pinned certificate

The pinned verifier binds the C-0102 parent and checks direct maximal minors, the two-by-two Newton tail, exhaustive low-dimensional cup incidence, matching stability, the symbolic hook recurrence and odd remainder, denominator divisibility, expanded shifted coordinates, and negative controls. Its bounded computations guard each algebraic link; the all-area conclusion comes from the symbolic recurrence and factor argument.

Run command

```
uv run --frozen python canon/witnesses/C-0104/verify.py
```

Pinned files

- `canon/witnesses/C-0104/PIN.md`
- `canon/witnesses/C-0104/verify.py`
- `canon/witnesses/C-0102`
- `canon/witnesses/C-0079`
- `scratch/adjacent-unconditional--half-shifted-two-row-hook-cup-all-area-positivity/RESULT.md`
- `canon/claims/C-0104-half-shifted-two-row-hook-cup-positivity-all-areas.md`

7 Scope

Coefficientwise half-shifted positivity is established for every two-row hook $(n, 1)$ with $d \geq n \geq 2$.

8 Sources

- Canonical claim: `canon/claims/C-0104-half-shifted-two-row-hook-cup-positivity-all-areas.md`
- Witness pin: `canon/witnesses/C-0104/PIN.md`
- Source result: `scratch/adjacent-unconditional--half-shifted-two-row-hook-cup-all-area-positivi RESULT.md`