One-row and one-column half-shifted cup positivity in every area

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Abstract

For the flagged complete-homogeneous array in dimension d, every gauged noncrossing-cup coordinate indexed by a one-row partition (n) or one-column partition (1^n) is coefficientwise nonnegative in $b = a - 1/2$. The proof reduces each boundary family to a two-incidence cup chain, evaluates the needed neighboring minors, derives cleared recurrences with nonnegative coefficients, and cancels every denominator against distinct factors of the principal minor.

$$\alpha_{(n)}, \alpha_{(1^n)} \in \mathbf{Q}_{\geq 0}[a - \tfrac{1}{2}] \qquad (1 \leq n \leq d)$$

1 Coordinates and the half shift

The starting array is $A_{k,j}(a) = (k+1)h_{2j-k-1}(a, a+1, \ldots, a+k+1)$. Its Catalan-admissible maximal minors H_λ are transformed by the inverse gauged noncrossing-incidence matrix into cup coordinates α_λ.

The theorem concerns coefficientwise positivity after the natural shift $b = a - 1/2$, which is stronger than merely showing that a coordinate is nonnegative at individual real values.

$$b = a - \frac{1}{2}$$

$$\alpha_{(n)}, \alpha_{(1^n)} \in \mathbf{Q}_{\geq 0}[b]$$

2 Two-incidence boundary chains

For a one-row partition, one isolated selected endpoint can pair only with an adjacent unselected endpoint. The two choices give the neighboring matchings indexed by (n) and $(n-1)$.

For a one-column partition, the analogous isolated unselected endpoint again has exactly two adjacent choices. After the canonical gauge, each row has incidences $+1$ and -1, so both cup coordinates become alternating sums of neighboring minors.

$$\alpha_{(n)} = \sum_{j=0}^{n} (-1)^j H_{(j)}$$

$$\alpha_{(1^n)} = \sum_{j=0}^{n} (-1)^j H_{(1^j)}$$

3 Closed neighboring-minor formulas

A unitriangular Newton-column change converts the flagged determinants into arithmetic divided differences. Replacing the final principal row gives the one-row ratio directly.

For one-column shapes, the remaining tail determinant is a binomial-Hessenberg cofactor of a proper Riordan array. Cofactor inversion followed by Lagrange inversion evaluates it in every size.

$$\frac{H_{(j)}}{H_\varnothing} = \frac{d+j}{d} \frac{\binom{d}{j}}{(Y)_j}$$

$$\frac{H_{(1^j)}}{H_\varnothing} = \frac{\binom{d+j}{j}}{(Y-j+1)_j}$$

$$Y = 2b + d + 1$$

4 Cleared recurrences

Set $m = d - n$ and clear the rising-factorial denominators from the alternating sums. The resulting bivariate polynomials Q_n and R_n satisfy first-order recurrences with an alternating final term.

Even indices are immediate positive induction steps. For odd indices, pairing two consecutive steps converts the apparent subtraction into an explicit remainder whose factors all have nonnegative coefficients in b and m.

$$Q_n = (2b + m + 2n) Q_{n-1}(m+1, b) + (-1)^n \frac{(m+2n)(m+1)_{n-1}}{n!}$$

$$R_n = (2b + m + 2) R_{n-1}(m+1, b) + (-1)^n \binom{m+2n}{n}$$

$$Q_n, R_n \in \mathbf{Q}_{\geq 0}[b, m]$$

1. Use $Q_1 = R_1 = 2b$ as the odd base.

2. Apply the one-step recurrence directly when n is even.
3. When $n \geq 3$ is odd, substitute the preceding even recurrence and factor the remaining term into nonnegative factors.

5 Returning to the original coordinates

The principal minor factors as $P_d(b) = d! \prod_{0 \leq p < q \leq d}(2b+p+q+1)$. Every factor in the cleared row or column denominator occurs at a distinct pair (p, q) in this product.

After cancellation, the remaining principal factor has strictly positive coefficients. Multiplying it by Q_n or R_n proves coefficientwise nonnegativity of the original cup coordinates.

$$P_d(b) = d! \prod_{0 \leq p < q \leq d} (2b + p + q + 1)$$

6 Pinned certificate

The pinned exact-arithmetic verifier binds the C-0086 parent and checks direct flagged determinants, Newton-tail and Riordan identities, exhaustive low-dimensional cup incidences, matching stability, the formal odd-step recurrences, denominator-factor witnesses, expanded shifted coordinates, and the exact boundary-zero classification. The finite checks guard the symbolic proof identities; they are not a finite-sampling substitute for the all-area argument.

Run command

```
uv run --frozen python canon/witnesses/C-0102/verify.py
```

Pinned files

- `canon/witnesses/C-0102/PIN.md`
- `canon/witnesses/C-0102/verify.py`
- `canon/witnesses/C-0086`
- `scratch/adjacent-unconditional--half-shifted-row-column-cup-all-area-positivity/proof.md`
- `scratch/adjacent-unconditional--half-shifted-row-column-cup-all-area-positivity/RESULT.md`
- `canon/claims/C-0102-half-shifted-row-column-cup-positivity-all-areas.md`

7 Scope

Coefficientwise half-shifted positivity is established for the one-row family (n) and one-column family (1^n) in every admissible area and dimension.

8 Sources

- Canonical claim: `canon/claims/C-0102-half-shifted-row-column-cup-positivity-all-areas.md`
- Witness pin: `canon/witnesses/C-0102/PIN.md`
- Proof source: `scratch/adjacent-unconditional--half-shifted-row-column-cup-all-area-positivity/`
  `proof.md`
- Source result: `scratch/adjacent-unconditional--half-shifted-row-column-cup-all-area-positivity`
  `RESULT.md`