The optimal nonnegative CHJ-II fixed-range Perron weight is constant

Kenan

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Abstract

In the $k = 1$ Cully-Hugill–Johnston II averaged Perron construction on $[T, 2T]$, the unique nonnegative normalized C^1 weight minimizing the source norm is the constant weight $w(u) = 1$. Its exact Perron coefficient is $(3 + 2\log 2)/\pi = 1.3962008\ldots$, replacing the published coefficient 1.785. The weighted theorem remains valid for complex coefficients, while the pointwise extraction is restricted to real-valued remainders.

$$\operatorname*{arg\,min}_{\substack{w\ge 0,\ w\in C^1[1,2] \\ \int_1^2 w=1}} N_{1,2}(w) = \{1\}, \qquad 4N_{1,2}(1) = \frac{3+2\log 2}{\pi}$$

1 The fixed-range optimization problem

For a nonnegative normalized C^1 weight on $[1, 2]$, the $k = 1$ source norm consists of two endpoint terms, a weighted integral of w, and the total-variation term weighted by $1/u$. The source imposes no endpoint-vanishing condition when $k = 1$, so the constant weight is admissible.

Direct substitution of $w = 1$ gives the exact norm and hence the exact Perron coefficient.

$$N_{1,2}(w) = \frac{1}{2\pi}\left(\frac{w(2)}{2} + w(1) + \int_1^2 \frac{w(u)}{u}\,du + \int_1^2 \frac{|w'(u)|}{u}\,du\right)$$

$$4N_{1,2}(1) = \frac{3+2\log 2}{\pi} < 1.397 < 1.785$$

2 Dual lower bound

Set $A = 3/2 + \log 2$ and $\phi(u) = 1 + \log u - A(u - 1)$. Elementary derivative signs give $|\phi(u)| \le 1/u$, with strict inequality in the interior, and the endpoint values match the two boundary coefficients in the norm.

Replacing $|w'|/u$ by $\phi w'$ and integrating by parts yields $2\pi N_{1,2}(w) \geq A \int_1^2 w = A$. Equality forces $w' = 0$ because the dual inequality is strict inside $(1, 2)$; normalization then forces $w = 1$.

$$\phi(1) = 1, \qquad \phi(2) = -\tfrac{1}{2}, \qquad |\phi(u)| \leq \frac{1}{u}$$

$$2\pi N_{1,2}(w) \geq A \int_1^2 w(u)\, du = A$$

1. Construct a dual function whose endpoint values reproduce the norm's boundary terms.
2. Prove the pointwise domination $|\phi(u)| \leq 1/u$.
3. Integrate by parts to obtain the universal lower bound and use strictness to identify the unique equality case.

3 Cutoff and source propagation

For the constant weight, the second source norm remains $3/(2\pi)$. The associated cutoff θ_{P} is the unique positive root of a cubic and is enclosed between 0.72068 and 0.72069.

Substituting the exact coefficient and the safe cutoff floor into the source's Section 4 formula strictly lowers all ten published M rows on their original domains. In the C-0090 specialization, the common-interval reserve falls from 3.543 to 3.263.

$$3\theta_{\mathrm{P}}^3 + 2\pi\theta_{\mathrm{P}}^2 - (3 + 2\log 2) = 0$$

$$0.72068 < \theta_{\mathrm{P}} < 0.72069$$

4 The real-valued pointwise step

The averaged Perron theorem itself remains valid for arbitrary complex coefficients. To infer a point $T^* \in [T, 2T]$ where the remainder has small modulus, however, continuity plus a nonnegative average is enough only when the remainder is real-valued.

If a continuous real remainder had modulus greater than the bound everywhere, it would keep one sign and force its weighted average to exceed the bound. The complex unit-circle function has average zero while retaining modulus one everywhere, so it is an exact counterexample to the unrestricted complex pointwise inference.

$$G(t) = \exp\left(2\pi i \frac{t - T}{T}\right), \qquad \int_T^{2T} G(t)\, dt = 0, \qquad |G(t)| = 1$$

5 What the certificate establishes

The 256-bit Arb verifier hash-pins the source archive, TeX, bibliography, supplementary notebook, and repository commit. It certifies the exact norm comparison, both dual inequalities, the cutoff root enclosure, all ten strict table improvements, and the C-0090 common-contour conditions.

The certificate also executes the complex-cancellation counterfeit. Its role is both positive and corrective: it verifies the improved constants and enforces the exact boundary between the complex averaged theorem and the real-valued pointwise corollary.

6 Pinned certificate

The verifier proves the exact fixed-range variational optimum, encloses the induced cutoff, propagates the coefficient through all ten source rows and the C-0090 specialization, and rejects the invalid complex pointwise inference.

Run command

```
uv run --frozen python canon/witnesses/C-0101/verify.py
```

Pinned files

- `canon/witnesses/C-0101/verify.py`
- `canon/witnesses/C-0101/counterfeit.py`
- `canon/witnesses/C-0101/PROOF.md`
- `canon/witnesses/C-0101/PIN.md`
- `canon/witnesses/C-0090/source/chj2-v3/main.tex`

7 Scope

The unique optimum is over nonnegative normalized C^1 weights in the fixed $k = 1$, $[T, 2T]$ construction. The pointwise corollary assumes a real-valued remainder.

8 Sources

- Canonical claim: `canon/claims/C-0101-chj-averaged-perron-constant-weight-optimum.md`
- Proof: `canon/witnesses/C-0101/PROOF.md`
- Source and verifier pin: `canon/witnesses/C-0101/PIN.md`

- Pinned CHJ-II source: `canon/witnesses/C-0090/source/chj2-v3/main.tex`