A smaller explicit threshold for decimal digit sums of primes

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Abstract

For every integer m at least 138033986655720044641496559943864 and coprime to nine, there is a prime $p \leq 10^{2m/9}$ whose decimal digit sum is m. The proof imports Lehmann's parameter-uniform analytic lemmas, replaces a coarse floor-error treatment by a signed one-sided envelope, certifies that every threshold condition remains valid thereafter, and performs the exact integer handoff from digit length to m.

$$m \geq 138033986655720044641496559943864,\ \gcd(m,9)=1 \implies \exists p \leq 10^{2m/9} : s_{10}(p) = m$$

1 The prescribed digit-sum problem

Write $s_{10}(n)$ for the sum of the decimal digits of n. Every integer is congruent to its digit sum modulo nine, so the condition $\gcd(m,9)=1$ is the natural congruence condition for seeking a prime with digit sum m.

The theorem is effective: it gives one explicit sufficient threshold and retains Lehmann's prime-size bound $p \leq 10^{2m/9}$.

$$n \equiv s_{10}(n) \pmod 9$$

$$m \geq M_{\text{new}},\ \gcd(m,9)=1 \implies \exists \text{ prime } p \leq 10^{2m/9} \text{ with } s_{10}(p) = m$$

2 The numerical improvement

Lehmann's pinned version proves the same statement with a larger explicit threshold. The new certificate keeps the theorem's domain, congruence condition, and exponent unchanged and lowers only the sufficient starting value.

No claim is made that either threshold is the least possible one.

$$M_{\text{old}} = 1778435903393816234231372922963 55$$

$$M_{\text{new}} = 13803398665572004464149655994 3864$$

$$M_{\text{old}} - M_{\text{new}} = 398096036836615787816407323524 91$$

3 Choosing a smaller permanent threshold

The specialization uses exact rational parameters $\eta = 545473/10000000$ and $\nu = 714321/2500000$, together with the source's $1/200$ safety factor for c_{43}. The pinned arithmetic checks that the parameters lie strictly inside all admissible inequalities.

A directed threshold $Y_\dagger = 7.063 \times 10^{31}$ is converted into an exact integer digit length $L_\dagger$. The proof then shows that each source condition holds not merely at that endpoint but for every integer $L \geq L_\dagger$.

$$Y_\dagger = 7.063 \times 10^{31}$$

$$L_\dagger = 30674219256826676586999235543080 = \left\lceil \frac{Y_\dagger}{\log 10} \right\rceil$$

1. Certify the moment-comparison conditions C1–C8 using monotone endpoint inequalities and exact handling of the relevant floors and residue classes.
2. Certify the characteristic-function conditions C9–C12 with decreasing continuous upper envelopes.
3. Check the major-arc, minor-arc, arithmetic-progression, absorption, and auxiliary integer thresholds with outward-directed margins.

4 The signed C13 floor envelope

The load-bearing improvement occurs in the second C13 inequality. With $\widetilde{D} = 2\theta L^\nu / \log L$ and $D = 2\lfloor \widetilde{D}/2 \rfloor$, the exact exponent is treated as a decreasing function of D on the entire interval allowed by the floor.

Because $\widetilde{D} - 2 < D \leq \widetilde{D}$, substituting $W = \widetilde{D} - 2$ gives a one-sided upper envelope rather than an absolute-value error bound. Its endpoint is strictly negative and its derivative remains negative, so C13 stays valid permanently.

$$h(D) = (D-1)T + \frac{D}{2} - \frac{D}{2} \log D$$

$$W = \widetilde{D} - 2 < D \leq \widetilde{D}$$

$$\frac{\partial h}{\partial D} = T - \frac{1}{2} \log D < 0$$

5 Exact handoff from digit length to m

Directed arithmetic proves the precise ceiling that defines M_{new}. Therefore every admissible m above the stated threshold has $L = \lfloor 2m/9 \rfloor \geq L_\dagger$.

All imported source hypotheses then hold at $x = 10^L$. Lehmann's positivity argument supplies a prime with the required digit sum, and the floor relation gives the advertised size bound.

$$M_{\text{new}} = \left\lceil \frac{9}{2} \frac{Y_\dagger}{\log 10} + \frac{9}{2} \right\rceil$$

$$L = \left\lfloor \frac{2m}{9} \right\rfloor \geq L_\dagger$$

$$p \leq 10^L \leq 10^{2m/9}$$

6 Pinned certificate

The pinned wrapper binds the Lehmann v1 source and local scripts, checks the source conditions with directed Arb arithmetic at three precisions, independently retypes the binding formulas in Julia/Nemo, and verifies the exact floor-and-ceiling conversion. The retained author-code replay is corroborative rather than load-bearing.

Run command

```
uv run --frozen python canon/witnesses/C-0092/verify_all.py
```

Pinned files

- `canon/witnesses/C-0092/PIN.md`
- `canon/witnesses/C-0092/PROOF.md`
- `canon/witnesses/C-0092/RELEASE-AUDIT.md`
- `canon/witnesses/C-0092/verify.py`
- `canon/witnesses/C-0092/verify_all.py`

- canon/witnesses/C-0092/independent_check.jl
- canon/witnesses/C-0092/source/lehmann-2606.04677v1/main.tex
- canon/claims/C-0092-lehmann-prime-digit-sum-surjectivity-threshold.md

7 Scope

The result certifies a specialization of the analytic lemmas in Lehmann arXiv:2606.04677v1 and lowers the explicit sufficient threshold by about 22.38%.

8 Sources

- Canonical claim: canon/claims/C-0092-lehmann-prime-digit-sum-surjectivity-threshold.md
- Witness pin: canon/witnesses/C-0092/PIN.md
- Proof artifact: canon/witnesses/C-0092/PROOF.md
- Release audit: canon/witnesses/C-0092/RELEASE-AUDIT.md
- Pinned source manuscript: canon/witnesses/C-0092/source/lehmann-2606.04677v1/main.tex