Sharp fixed-template bounds for the pole-cancelled zeta derivative

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Abstract

Let $Z(s) = \zeta'(s) + (s-1)^{-2}$ and $\alpha = -\gamma_1$, where γ_1 is the first Stieltjes constant. The result proves $\alpha/s^2 < Z(s) < 151\alpha/(151 + (s-1)^2)$ for every real $s > 1$. The numerator constants are best possible for those two fixed denominator templates, and directed decimal rounding gives a strict improvement over the displayed simple constants 1/14 and 11 in the pinned source.

$$\frac{-\gamma_1}{s^2} < \zeta'(s) + \frac{1}{(s-1)^2} < \frac{-151\gamma_1}{151 + (s-1)^2} \quad (s > 1)$$

1 The pole-cancelled quantity

The derivative $\zeta'(s)$ has a double-pole term $-(s-1)^{-2}$ at $s = 1$. Adding $(s-1)^{-2}$ removes that singular term and leaves a finite positive quantity on $s > 1$.

With $t = s-1$ and the Stieltjes expansion, the cancellation produces a convergent local series whose value tends to $\alpha = -\gamma_1$ as $t \to 0^+$.

$$Z(1+t) = \zeta'(1+t) + \frac{1}{t^2} = \sum_{n=1}^{\infty} \frac{(-1)^n \gamma_n}{(n-1)!} t^{n-1}$$

$$\lim_{t \to 0^+} Z(1+t) = \alpha$$

2 The two global inequalities

The lower envelope uses the fixed denominator $s^2 = (1+t)^2$. The upper envelope uses the fixed denominator $151 + t^2$. The theorem selects the exact admissible numerator at the boundary $t = 0$ for each template.

Both comparisons are strict for every $t > 0$, while their limiting equality at $t = 0$ is what later forces sharpness.

$$\frac{\alpha}{(1+t)^2} < Z(1+t) < \frac{151\alpha}{151 + t^2} \qquad (t > 0)$$

3 A proof split across all positive t

The direct proof covers three adjoining ranges. Near the pole it uses a rigorous Laurent remainder and positivity of Bernstein coefficients. On a compact middle interval it uses convex-quadrature bounds and a gap-free interval cover. On the infinite tail it reduces both comparisons to monotonic estimates.

The source theorem supplies stronger rational envelopes, but the retained direct proof does not use that theorem or its undisplayed computations as theorem inputs.

$$(0, \tfrac{1}{2}] \;\cup\; [\tfrac{1}{2}, 4] \;\cup\; [4, \infty) = (0, \infty)$$

1. For $0 < t \le 1/2$, truncate the Stieltjes series and bound the remainder by $E_N(t) = 4\sqrt{2}\, c^{N+1} t^N/(1-ct)$ with $c = 2/(\pi e)$.
2. After clearing positive denominators, certify every Bernstein coefficient of the lower degree-five and upper degree-seven comparison polynomials as positive.
3. For $1/2 \le t \le 4$, place $Z(1+t)$ between explicit convex midpoint and trapezoid bounds $A_7(t)$ and $B_{83}(t)$, then certify both target gaps over 16,384 exact rational cells.
4. For $t \ge 4$, use decreasing correction terms in A_7 for the lower bound and $B_{83}(t) < t^{-2}$ together with a positive constant margin for the upper bound.

4 Why the constants are sharp

Suppose a numerator C works in the lower template C/s^2. Sending $s \to 1^+$ makes the denominator tend to one and $Z(s)$ tend to α, so necessarily $C \le \alpha$.

Likewise, if $D/(151+(s-1)^2)$ is a valid upper template, the same limit gives $\alpha \le D/151$, hence $D \ge 151\alpha$. The proved global inequalities show that these boundary values themselves are admissible.

$$\sup\left\{C : \frac{C}{s^2} < Z(s)\ \forall s > 1\right\} = \alpha$$

$$\inf\left\{D : Z(s) < \frac{D}{151 + (s-1)^2}\ \forall s > 1\right\} = 151\alpha$$

5 Directed decimal corollary

Outward-directed comparisons place a terminating lower decimal strictly below α and a terminating upper decimal strictly above 151α. This produces a readily usable rational-looking corollary without weakening strictness.

On the same domain and with the same denominators, these decimals improve the source's displayed simple constants 1/14 and 11.

$$\frac{1}{14} < 0.0728158 < \alpha$$

$$151\alpha < 10.995193 < 11$$

$$\frac{0.0728158}{s^2} < Z(s) < \frac{10.995193}{151 + (s-1)^2}$$

6 Pinned certificate

The pinned package binds the exact statement and source bytes, checks the Laurent remainder, every small-range Bernstein coefficient, all 16,384 middle-range cells, both monotone tails, sharp-limit arguments, and outward decimal rounding. It also includes an independent PARI/GP reconstruction, while the primary verifier remains sufficient on its own.

Run command

```
uv run --frozen python canon/witnesses/C-0091/verify.py
```

Pinned files

- `canon/witnesses/C-0091/PIN.md`
- `canon/witnesses/C-0091/RESULT.md`
- `canon/witnesses/C-0091/theorem.json`
- `canon/witnesses/C-0091/verify.py`
- `canon/witnesses/C-0091/run_independent.py`
- `canon/witnesses/C-0091/independent_check.gp`
- `canon/witnesses/C-0091/source/arxiv/remarque.tex`
- `canon/claims/C-0091-sharp-pole-cancelled-zeta-derivative-bounds.md`

7 Scope

Sharpness is established for the two displayed fixed denominator templates for $Z(s) = \zeta'(s) + (s-1)^{-2}$ on real $s > 1$.

8 Sources

- Canonical claim: `canon/claims/C-0091-sharp-pole-cancelled-zeta-derivative-bounds.md`
- Witness pin: `canon/witnesses/C-0091/PIN.md`
- Direct proof artifact: `canon/witnesses/C-0091/RESULT.md`
- Pinned primary source: `canon/witnesses/C-0091/source/arxiv/remarque.tex`