Complete area-at-most-six half-shifted cup positivity in every rank

Kenan

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Abstract

For every rank, every available gauged noncrossing-cup coordinate indexed by a partition of area at most six is coefficientwise nonnegative in $b = a - 1/2$ and is positive for $a > 1/2$. The proof extends the previously established area-at-most-five sector by evaluating all eleven area-six neighboring minors, inserting them into a rank-stable leading block of the inverse cup-incidence transform, cancelling every denominator content factor against the principal product, and reducing each new coordinate to a shifted sextic whose coefficients are positive polynomials in the excess rank. The certificate checks the all-rank symbolic determinant and sextic identities and uses finite exact reconstructions as implementation and stability guards.

$$r \geq 2,\ |\lambda| \leq 6,\ \lambda \text{ available} \Longrightarrow \alpha_\lambda(a) \in \mathbf{Q}_{\geq 0}[a - \tfrac{1}{2}]$$

1 What is proved

Put $d = r-1$ and $b = a-1/2$. A partition λ determines an admissible row set K_λ and a flagged minor $H_\lambda(a)$. The associated cup coordinate is obtained from the gauged inverse noncrossing-incidence transform.

For every rank and every partition of area at most six that is actually available in that rank, the coordinate polynomial has nonnegative rational coefficients in b. The availability qualifier is essential: among the eleven area-six shapes, one occurs at $d = 3$, seven at $d = 4$, nine at $d = 5$, and all eleven at $d \geq 6$.

$$A_{k,j}(a) = (k+1)h_{2j-k-1}(a, a+1, \ldots, a+k+1)$$

$$K_\lambda = (i - 1 + \lambda_{d+1-i})_{i=1}^{d}, \qquad H_\lambda(a) = \det A[K_\lambda, [d]]$$

$$\alpha_\lambda(a) \in \mathbf{Q}_{\geq 0}[a - \tfrac{1}{2}] \qquad (|\lambda| \leq 6)$$

2 Newton-tail determinant reduction

The flagged minors are converted into nested divided differences. After a Newton column change and cancellation of the common principal prefix, each area-six neighboring minor becomes a determinant of a small normalized tail matrix.

The master finite-difference identity gives every tail entry as a rational rising-factorial expression. Bareiss evaluation of determinants of sizes one through six yields the eleven explicit area-six ratios, including the rank polynomials $d^2 + d - 10$ and two quartic factors.

$$L_{e+t}F_e = \begin{cases} \binom{e}{t}(2a+e+t)_{e-t}, & 0 \le t \le e, \\ 0, & t > e, \end{cases}$$

$$\frac{L_{d+s}F_{d-c}}{L_{d-c}F_{d-c}} = \frac{\binom{d-c}{c+s}}{(Y-c)_{c+s}}, \qquad Y = 2a + d$$

3 Rank-stable cup transform

Order the thirty partitions of area at most six by stable area order. Relative to the area-at-most-five block, the cup-incidence matrix is block lower unitriangular with an eleven-row area-six extension. Its inverse gauged transform therefore gives each new coordinate as one explicit signed combination of lower-area and area-six minors.

When the rank increases, every available row set gains a leading zero and every matching gains the outer arc $(0, 2d+1)$. The raw cup sign and the global gauge both change sign, so their product is stable. Area monotonicity makes the full incidence matrix lower unitriangular, which prevents later-area matching columns from entering the leading block.

$$W_{\le 6} = \begin{pmatrix} W_{\le 5} & 0 \\ B_6 & I_{11} \end{pmatrix}, \qquad W_{\le 6}^{-1} S_{\le 6} = \begin{pmatrix} T_{\le 5} & 0 \\ C_6 & I_{11} \end{pmatrix}$$

$$K_\lambda^{(d)} = (0, 1 + K_\lambda^{(d-1)})$$

4 Shifted sextics and positivity

For each area-six shape, write $m = d - d_{\min}(\lambda)$. Substitution of the eleven minor ratios into the stable transform reduces the coordinate to the principal product times a sextic numerator divided by six content factors.

Every coefficient of each numerator is a strictly positive polynomial in $m \ge 0$. Every denominator content factor occurs as a pair-sum factor of the principal product, including repeated positive, zero, and negative contents. After cancellation, the remaining pair product also has strictly positive coefficients. This proves strict coefficient positivity for every new area-six coordinate and, together with the parent area-five theorem, coefficientwise nonnegativity for the complete area-at-most-six sector.

$$P_d(b) = d! \prod_{0 \le p < q \le d} (2b + p + q + 1)$$

$$\alpha_\lambda(a) = \frac{N_\lambda(b, m)}{c_\lambda \prod_{u \in \lambda}(X + \mathrm{ct}(u))} P_d(b), \qquad X = 2b + d + 1$$

$$N_\lambda(b, m) \in \mathbf{Z}_{>0}[b, m]$$

5 Boundary values and proof route

Every coordinate is strictly positive for $a > 1/2$. At $a = 1/2$, the only zeros in the complete area-at-most-six sector are the area-one coordinate in each rank and the coordinate indexed by $(1, 1, 1)$ when $d = 3$.

The proof route is all-rank algebra rather than finite extrapolation: derive symbolic tail determinants, prove the stable cup rows and later-column exclusion, obtain symbolic sextic identities, shift the rank by the true minimum dimension, and read positivity from every bivariate coefficient. Finite direct determinants and cup blocks are consistency checks for these formulas.

$$\alpha_\lambda(a) > 0 \quad (a > \tfrac{1}{2})$$

$$\alpha_\lambda(\tfrac{1}{2}) = 0 \quad \text{only for the stated area-one family and } (1, 1, 1) \text{ at } d = 3$$

1. Evaluate the eleven neighboring-minor ratios from the normalized Newton tails.
2. Apply the stable thirty-row inverse cup transform.
3. Cancel every content denominator against an explicit principal pair factor.
4. Prove every coefficient of each shifted sextic is positive for $m \ge 0$.

6 Certificate role

The pinned verifier checks the parent C-0084 hash, 117 direct flagged-minor and coordinate comparisons, all eleven symbolic tail determinant identities, all eleven symbolic shifted sextic identities, all-rank bivariate coefficient positivity, content-factor witnesses, primitivity, coprimality, cup-block reconstruction, rank-stability regressions, and transcript digests.

The determinant, sextic, and positivity checks are symbolic in the rank variable. The executable exercises outer-arc stability through dimension thirty and later-column exclusion through finite dimensions as regression guards; the all-rank extension of those cup facts is the explicit outer-arc and area-order argument in the canonical proof, not an inference from the finite checks.

7 Pinned certificate

The certificate verifies the eleven area-six tail determinants and shifted sextics symbolically in the rank variable, proves their coefficient signs and denominator cancellations, and checks the finite implementations and cup-stability machinery against pinned transcripts. The written outer-arc and area-order arguments supply the remaining all-rank cup-block logic.

Run command

```
uv run --frozen python canon/witnesses/C-0086/verify.py
```

Pinned files

- `canon/witnesses/C-0086/PIN.md`
- `canon/witnesses/C-0086/verify.py`
- `canon/witnesses/C-0086/INDEPENDENT-REVIEW.md`
- `canon/witnesses/C-0084/verify.py`

8 Scope

The theorem covers every available gauged noncrossing-cup coordinate indexed by a partition of area at most six, in every rank.

9 Sources

- Canonical claim: `canon/claims/C-0086-half-shifted-area-six-cup-positivity-all-ranks.md`
- Witness pin: `canon/witnesses/C-0086/PIN.md`
- Proof result: `scratch/adjacent-unconditional--half-shifted-area-six-cup-all-rank-factorization RESULT.md`
- Independent review: `canon/witnesses/C-0086/INDEPENDENT-REVIEW.md`
- Prior-art boundary: `literature/2026-07-19-half-shifted-area-six-cup-all-rank-search.md`
- Pinned proof source: `canon/witnesses/C-0086/verify.py`