Axler's Table 9 coefficient-1.149 row fails at its endpoint and has a sharp repair

Kenan

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Abstract

Axler's coefficient-1.149 bound for $\pi(x)$ fails at its published composite endpoint $x_0 = 42\,575\,222\,481$. The exact same-range non-strict coefficient threshold is $1.1490003091852194519\ldots$, so 1.14900031 gives a strict repair with the same least integer start. If the printed coefficient 1.149 is retained, the least valid integer start is 42 575 222 505.

$$\pi(x) < \frac{x}{\log x - 1 - 1.14900031/\log x} \quad (x \geq 42\,575\,222\,481)$$

1 What is proved

At $x_0 = 42\,575\,222\,481$, the exact count is $\pi(x_0) = 1\,817\,311\,115$, and the published strict inequality has its sides in the wrong order. Defining $a_* = \log x_0(\log x_0 - 1 - x_0/\pi(x_0))$ gives the exact threshold on the entire published half-line.

Equality for the non-strict bound with coefficient a_* occurs only at x_0. Every larger admissible coefficient gives the strict bound from x_0, while the decimal 1.14900031 is small enough that $x_0 - 1$ still fails, preserving minimality of the published start.

$$a_* = 1.149000309185219451903089860600 8869\ldots$$

$$\pi(x_0) - \frac{x_0}{\log x_0 - 1 - 1.149/\log x_0} = 0.9799543748\ldots > 0$$

2 Normalize the inequality

On the positive-denominator domain, the prime-counting inequality is equivalent to a comparison with $B(x) = \log x(\log x - 1 - x/\pi(x))$. Between consecutive primes, $\pi(x)$ is constant and $B(x)$ is strictly decreasing for $x \geq 9$. Therefore, after the initial endpoint, only prime points can create a new maximum.

This reduction turns a real-variable statement into an endpoint check, a finite list of prime candidates, and an analytic tail.

$$\pi(x) < \frac{x}{\log x - 1 - c/\log x} \quad \Longleftrightarrow \quad B(x) < c$$

$$B'(x) \le -\frac{3}{x} < 0 \quad \text{between prime jumps}$$

3 Exhaust the finite range

The pinned scanner covers every prime between x_0 and $H = 55\,576\,240\,054$, a total of $528\,206\,511$ prime candidates. It bounds each logarithm from above with directed fixed-point arithmetic and converts $B(p) < 1.149$ into an exact integer inequality.

The least finite-range reserve occurs at the first later prime $42\,575\,222\,531$ and remains positive. Segment counts are reconciled with pinned prime-count computations, and the arithmetic-width bounds show that the scanner's signed and unsigned integer operations cannot overflow.

$$1149 \cdot 2^{56} - 1000U > 0$$

$$1.149 - B(42\,575\,222\,531) = 1.9879406218\ldots \times 10^{-8}$$

4 Close the infinite tail

At the finite handoff, the proof compares the target rational function with $\mathrm{li}(x)$. A derivative polynomial shows that this difference is increasing from H, while Buthe's verified inequality $\mathrm{li}(x) > \pi(x)$ covers the range through 10^{19}.

Beyond 10^{19}, Dusart's explicit upper bound for $\pi(x)$ reduces the target inequality to positivity of a cubic polynomial in $\log x$. Its value and first three derivatives have the required signs at the handoff, so positivity persists for the full tail.

$$F_c(x) = \frac{x}{\log x - 1 - c/\log x} - \mathrm{li}(x)$$

$$(\log x)^3 D_c(x)^2 F_c'(x) = (c-1)(\log x)^2 - 3c\log x - c^2$$

5 Locate the sharp repairs

The endpoint, finite scan, and analytic continuation prove $B(x) < a_*$ for every $x > x_0$. This establishes the exact same-range coefficient classification, not merely a rounded improvement.

For coefficient 1.149, the initial prime-free plateau has a unique real crossing at $42\,575\,222\,504.9836\ldots$. Hence the inequality fails at integer $42\,575\,222\,504$ and holds for every real $x \geq 42\,575\,222\,505$.

$$1.14900031 - a_* = 8.14780548\ldots \times 10^{-10} > 0$$

$$N_{\min}(1.149) = 42\,575\,222\,505$$

6 Role of the certificate

The public command runs the pinned independent reconstruction. It checks the decisive counts and endpoints, reconstructs all finite-tail partition boundaries with two prime-counting algorithms, and re-evaluates the analytic margins without rerunning the primary exhaustive prime scanner.

The primary proof package separately contains the full scanner and directed Arb verification. The disclosed residual limitation is that the 528 206 511-prime interior has one complete prime-enumeration implementation, although its partition counts and fixed-point inequalities receive independent checks.

7 Pinned certificate

The public independent checker reconstructs the endpoint failure, sharp coefficient, corrected start, all partition-boundary counts, and the analytic handoffs. It corroborates the pinned primary scanner while explicitly retaining the one-enumerator caveat for the full finite interior.

Run command

```
uv run --frozen python canon/witnesses/C-0081/independent_check.py
```

Pinned files

- `canon/witnesses/C-0081/independent_check.py`
- `canon/witnesses/C-0081/tail-output.txt`
- `canon/witnesses/C-0081/source`

8 Scope

The correction applies to Axler's coefficient-1.149 row in Proposition 11 and Table 9, on the positive-denominator domain and the published starting range.

9 Sources

- Canonical claim: `canon/claims/C-0081-axler-table9-1149-counterexample-sharp-repair.md`
- Certificate pin: `canon/witnesses/C-0081/PIN.md`
- Independent review: `canon/witnesses/C-0081/INDEPENDENT-REVIEW.md`