All half-shifted flagged cup coordinates are positive through rank eight

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Abstract

For ranks two through eight, every gauged noncrossing-cup coordinate of the flagged array $A_{k,j}(a) = (k+1)h_{2j-k-1}(a, a+1, \ldots, a+k+1)$ is coefficientwise nonnegative in the half-shift variable $b = a - 1/2$. The proof factors the array through a half-shifted finite-difference matrix and a Pascal translation matrix. Cauchy-Binet reduces every shifted coefficient to an entry of an exact cup-transformed compound matrix B_d, which is reconstructed through dimension seven. Every positive-degree coefficient is strictly positive, and the exceptional zero constant terms are classified. Separately, an explicit negative leading 2×2 minor shows that this kernel is not totally nonnegative in any dimension.

$$2 \le r \le 8 \Longrightarrow \alpha_M(a) \in \mathbf{Q}_{\ge 0}[a - \tfrac{1}{2}]$$

1 Coordinates and indexing

Put $d = r - 1$. Catalan-admissible row sets $K = (k_1 < \cdots < k_d)$ index maximal flagged minors $H_K(a)$. After lifting K to a ballot subset, noncrossing perfect matchings index the cup basis.

With the specified area order and global gauge, the cup-incidence matrix W_d is lower unitriangular. The cup coordinates are obtained by applying its inverse to the checkerboard-signed minor vector.

$$A_{k,j}(a) = (k+1)h_{2j-k-1}(a, a+1, \ldots, a+k+1)$$

$$H_K(a) = \det A[K, [d]], \qquad \alpha(a) = W_d^{-1} S_d(H_K(a))_K$$

2 Half-shift factorization

The divided-difference identity identifies each flagged entry with a normalized forward difference of an even monomial. After writing $a = b + 1/2$, translation separates the arithmetic half-shift from a fixed finite-difference matrix.

Cauchy-Binet then expands every shifted coordinate over Catalan staircase subsets. Each supported Pascal minor is a positive constant times a single monomial in b, while every unsupported minor vanishes. Therefore coefficientwise positivity reduces exactly to entrywise nonnegativity of the transformed compound matrix B_d.

$$A_{k,j}(a) = \frac{1}{k!}\Delta^{k+1}x^{2j}\big|_{x=a}$$

$$A(b+\tfrac{1}{2}) = FP_d(b), \qquad P_d(b)_{n,j} = \binom{2j}{n} b^{2j-n}$$

$$\mathrm{B}_d = W_d^{-1} S_d C_d(F)|_{\mathcal{A}_d \times \mathcal{A}_d}$$

3 Finite positivity through rank eight

The exact reconstruction proves $\mathrm{B}_d \geq 0$ entrywise for $1 \leq d \leq 7$, corresponding to ranks $2 \leq r \leq 8$. At $d = 7$, all $1430^2 = 2{,}044{,}900$ entries are determined exactly: 2,044,899 are positive and one is zero.

Combining these signs with the positive Pascal minors proves that every coordinate polynomial is nonzero and coefficientwise nonnegative in b. Every coefficient of positive degree is strictly positive. The constant term vanishes only for the matching with left endpoints $\{0, 1, \ldots, d-1, d+1\}$ and, only at $d = 3$, for the matching with left endpoints $\{0, 2, 3, 4\}$.

$$\alpha_M(b+\tfrac{1}{2}) = \sum_{N \in \mathcal{A}_d} \mathrm{B}_d(M, N) \det\left[\binom{2j}{n_i}\right]_{i,j=1}^{d} b^{d(d+1)-|N|}$$

$$\alpha_M(a) > 0 \quad (a > \tfrac{1}{2}), \qquad \alpha_M(\tfrac{1}{2}) \geq 0$$

4 Sharp threshold and kernel obstruction

Rank two already gives coordinates $2a + 1$ and $2a - 1$. Thus $a = 1/2$ is the sharp lower endpoint for simultaneous nonnegativity of every coordinate.

The same transformed kernel cannot be totally nonnegative. Its leading 2×2 minor is negative in every dimension. Consequently it cannot be a planar path matrix or a product of nonnegative Jacobi chips, even though its entries are nonnegative through dimension seven.

Before the arithmetic specialization, the smallest cancelling entry is $x_0 + x_1 - 2$, which is not generically monomial- or key-nonnegative. It becomes $2b$ only at the exact common half-shift and unit spacing.

$$\alpha_{\text{nested}} = 2a + 1, \qquad \alpha_{\text{adjacent}} = 2a - 1$$

$$\det \mathrm{B}_d[\{0,1\},\{0,1\}] = -\frac{d+1}{d}(d!)^2 < 0$$

5 Proof route

The route is exact: reconstruct Catalan labels and cup signs, derive the finite-difference factorization, use Cauchy-Binet and the Pascal support criterion, compute the restricted compound of the half-shift matrix, solve the lower-unitriangular cup system, and collect the monomial powers of b.

A direct bypass through rank five evaluates the original flagged determinants on an exact interpolation grid and recovers the same coordinate polynomials without assuming the factorized conclusion. The orientation counterfeit omits the global gauge and makes the expected sign counts and positivity fail.

1. Generate the cup-incidence matrix with the selected-right-endpoint orientation.
2. Compute the exact half-shift compound and solve $W_d \mathrm{B}_d = S_d C_d(F)$.
3. Combine each nonnegative kernel entry with its positive Pascal monomial.
4. Compare with the direct exact interpolation route at low rank.

6 Certificate role

The pinned certificate reconstructs all 2,248,354 restricted compound minors through $d = 7$, verifies the complete kernel sign counts and zero labels, proves the strict coefficient and boundary-zero pattern, checks direct interpolation through rank five, and reproduces the pinned transcripts.

It also verifies the symbolic leading-block formula and scans it through dimension sixty-four as a mutation guard. The all-dimension negative-minor conclusion comes from the displayed symbolic algebra; the certificate does not establish entrywise nonnegativity of B_d beyond $d = 7$.

7 Pinned certificate

The certificate exactly reconstructs the half-shift kernel and every shifted coordinate through rank eight, independently checks the original minors through rank five, classifies the only zero constants, and proves the separate all-rank negative leading-minor identity. It does not certify arbitrary-rank entrywise positivity.

Run command

```
uv run --frozen python canon/witnesses/C-0071/verify.py
```

Pinned files

- `canon/witnesses/C-0071/PIN.md`
- `canon/witnesses/C-0071/verify.py`
- `canon/witnesses/C-0071/results.json`
- `canon/witnesses/C-0071/counterfeit-results.json`
- `canon/witnesses/C-0071/source/pre-audit-sympy-verify.py`

8 Scope

Complete coordinate positivity and coefficient-support classification are established through rank eight; the displayed negative leading minor is established in every rank.

9 Sources

- Canonical claim: `canon/claims/C-0071-half-shifted-stirling-cup-positivity-through-rank-eight.md`
- Witness pin: `canon/witnesses/C-0071/PIN.md`
- Proof result: `scratch/adjacent-unconditional--flagged-jacobi-trudi-temperley-lieb-cup-intertwi RESULT.md`
- Certificate audit: `scratch/proof-system--c0071-half-shifted-cup-source-certificate-audit/RESULT.md`
- Prior-art boundary: `literature/2026-07-19-half-shifted-stirling-cup-kernel-search.md`
- Pre-audit proof source: `canon/witnesses/C-0071/source/pre-audit-sympy-verify.py`
- Pinned proof source: `canon/witnesses/C-0071/verify.py`