Degree seven is the sharp all-rank Karlin polynomial threshold

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Abstract

For a positive-leading real polynomial f of degree at most six with all zeros nonpositive, every signed centered consecutive-derivative determinant $K_m(f;x)$ is nonnegative for every rank and every $x \geq 0$. The proof exhausts the ranks: direct positive root expansions handle ranks one through three, three pinned theorems handle ranks four through six, a unique anti-triangular permutation gives strict rank-seven positivity in degree six, and a zero-column argument gives all higher ranks. Degree seven is minimal for failure because an exact rank-six counterexample exists, including one with simple strictly negative zeros.

$$\mathrm{lc}(f) > 0,\ \deg f \leq 6,\ Z(f) \subseteq (-\infty, 0] \Longrightarrow K_m(f;x) \geq 0$$

1 Statement and sign convention

For each rank $m \geq 1$, the determinant is normalized by the Karlin sign. The positive-leading hypothesis is part of the theorem because an odd-rank determinant scales by an odd power of the polynomial.

The result covers every $x \geq 0$ and every rank simultaneously for the full class of degree-at-most-six real polynomials with zeros in $(-\infty, 0]$.

$$K_m(f;x) = (-1)^{m(m-1)/2} \det[f^{(i+j)}(x)]_{i,j=0}^{m-1}$$

$$\deg f \leq 6, \quad Z(f) \subseteq (-\infty, 0], \quad \mathrm{lc}(f) > 0 \quad \Longrightarrow \quad K_m(f;x) \geq 0$$

2 Ranks one through three

For $x > 0$, write $f(t) = a \prod_\nu (t + r_\nu)$ with $a > 0$ and $r_\nu \geq 0$, and set $\alpha_\nu = (x + r_\nu)^{-1}$. The normalized Taylor expansion of $f(x + z)$ converts derivatives into elementary symmetric functions of the positive numbers α_ν.

The rank-two expression is the positive power sum p_2. The rank-three determinant has a complete monomial expansion with coefficients 8 and 12, so it is nonnegative. Continuity extends these identities to $x = 0$, including cases with roots at zero.

$$\frac{f(x+z)}{f(x)} = \prod_{\nu}(1+\alpha_\nu z)$$

$$\frac{K_2(f;x)}{f(x)^2} = p_2$$

$$\frac{K_3(f;x)}{f(x)^3} = 8\sum_{\mu<\nu}\alpha_\mu^3\alpha_\nu^3 + 12\sum_{\lambda<\mu<\nu}\alpha_\lambda^2\alpha_\mu^2\alpha_\nu^2$$

3 Ranks four through six

The middle ranks are supplied by three exact canonical results using the same sign convention. Rank four is nonnegative through degree twelve, rank five through degree seven under the positive-leading restriction, and signed rank six through degree six.

Because the present domain has degree at most six, these three bounds cover every polynomial under consideration without an interpolation or genericity assumption.

1. Apply C-0043 to obtain $K_4(f;x) \geq 0$.
2. Apply C-0056 to obtain $K_5(f;x) \geq 0$.
3. Apply C-0059 to obtain $K_6(f;x) \geq 0$.

4 Rank seven and all higher ranks

If $\deg f = 6$ with leading coefficient $a > 0$, an entry of the 7×7 derivative Hankel matrix vanishes whenever $i+j > 6$. Any nonzero determinant term must satisfy $i+\sigma(i) \leq 6$ for every row. Summing these inequalities forces equality everywhere, so reversal is the unique contributing permutation.

The reversal sign cancels the Karlin sign, leaving a positive seventh power. If $\deg f \leq 5$, rank seven already has a zero last column. For every $m \geq 8$ and every degree at most six, the last column starts with derivative order $m - 1 \geq 7$, so the determinant is zero.

$$K_7(f;x) = (6!a)^7 = (720a)^7 > 0 \qquad (\deg f = 6)$$

$$K_m(f;x) = 0 \qquad (m \geq 8)$$

5 Why degree seven is sharp

A boundary degree-seven polynomial already fails at rank six. To remove repeated roots and roots at zero, the proof gives seven pairwise distinct positive reciprocal roots obtained by a rational perturbation of four roots near 53 and three roots near 1.

The corresponding product polynomial has simple strictly negative zeros, positive leading coefficient, and an exact negative signed rank-six determinant. Thus degree seven is the least degree at which any rank can fail; the statement does not say that every rank fails in degree seven.

$$f(x) = x^4(x+1)^3, \qquad K_6(f;0) = -1246946918400$$

$$\left(\frac{53q-3}{q}, \frac{53q-1}{q}, \frac{53q+1}{q}, \frac{53q+3}{q}, \frac{q-2}{q}, 1, \frac{q+2}{q}\right), \qquad q = 10^7$$

6 Pinned certificate

The certificate verifies the exhaustive rank decomposition, pins and reruns the rank-four through rank-six dependencies, and checks the degree-seven sharpness witnesses exactly. Its role is to certify that every rank is covered with one consistent sign convention and that failure first occurs in degree seven.

Run command

```
uv run --frozen python canon/witnesses/C-0065/verify.py
```

Pinned files

- `canon/witnesses/C-0065/PIN.md`
- `canon/witnesses/C-0065/verify.py`
- `canon/witnesses/C-0065/source/original-verify.py`
- `canon/witnesses/C-0065/source/craven-csordas-2002-iterated-laguerre-turan.txt`
- `canon/witnesses/C-0065/source/craven-csordas-2005-karlin-conjecture.txt`

7 Scope

The degree-seven threshold is sharp for positive-leading real polynomials with nonpositive zeros and the signed centered consecutive-derivative determinant family.

8 Sources

- Canonical claim: `canon/claims/C-0065-sharp-all-rank-karlin-degree-threshold.md`
- Witness pin: `canon/witnesses/C-0065/PIN.md`
- Proof result: `scratch/adjacent-unconditional--karlin-all-order-sharp-degree-six/RESULT.md`
- Certificate audit: `scratch/proof-system--c0065-all-rank-karlin-synthesis-certificate-audit/RESULT.md`
- Prior-art boundary: `literature/2026-07-19-karlin-all-order-sharp-degree-search.md`
- Pinned proof source: `canon/witnesses/C-0065/verify.py`