# Axler's global fourth- and fifth-order prime intervals improve sharply

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**Abstract**

For every real $x \geq 2$ and $n \in \{4, 5\}$, there is a prime in $(x, x(1 + B_n/\log^n x)]$ with $B_4 = 68.499418523333785\ldots$ and $B_5 = 126484.607970195861\ldots$. The order-four value is the actual least full-range coefficient, forced by the prime gap $1327 < 1361$. The order-five value is exact for the corrected Fiori–Kadiri–Swidinsky step-envelope splice, but is not claimed globally least for the primes.

$$x < p \leq x\left(1 + \frac{B_n}{\log^n x}\right), \quad n \in \{4, 5\}, \quad B_4 = \frac{34}{1327}\log^4(1327),\ B_5 = \frac{252969215940000000000}{1999999999996903}$$

## 1 What is proved

The theorem improves Axler's fourth- and fifth-order prime intervals on the identical range $x \geq 2$. The interval is open at $x$ and closed at its right endpoint.

$$B_4 = \frac{34}{1327}\log^4(1327) = 68.49941852333337850616\ldots$$

$$B_5 = \frac{252969215940000000000}{1999999999996903} = 126484.6079701958614154\ldots$$

$$x < p \leq x\left(1 + \frac{B_n}{\log^n x}\right), \qquad n \in \{4, 5\}$$

## 2 The sharp order-four gap

For $G_n(x) = x(1 + B_n/\log^n x)$, the order-four derivative has one low critical point inside the gap $11 < x < 13$. The certificate proves $G_4(x) > 13$ there and strict increase afterward.

An exhaustive sieve through 17,051,887 compares every consecutive prime gap. The unique largest requirement occurs at $1327 < 1361$; the runner-up $113 < 127$ is lower by more than 6.6216.

$$G_n'(x) = 1 + B_n \frac{\log x - n}{(\log x)^{n+1}}$$

$$\max_{p<q \text{ consecutive}} \frac{q-p}{p} \log^4 p = \frac{34}{1327} \log^4(1327) = B_4$$

# 3 Why B4 is globally least

At $x = 1327$, the right endpoint is exactly 1361, so the proposed interval reaches the next prime with equality. Any smaller coefficient leaves the entire interval below 1361.

This single prime-free interval proves actual full-range optimality for $B_4$, independently of the later analytic splice.

$$1327 \left(1 + \frac{B_4}{\log^4(1327)}\right) = 1361$$

# 4 Bridges and numerical cells

Axler's order-three theorem carries the order-four interval from the finite sieve into the corrected Fiori–Kadiri–Swidinsky range. Once order four is established, it similarly carries order five into that range.

For a two-sided theta-error row $(L_i, \epsilon_i)$ and $y = x + B_n x/L^n$, positivity of $\theta(y) - \theta(x)$ follows from an exact cell inequality. The first Table 2 theta exponent is repaired from Proposition 17 before all 236 cells are compared.

$$\theta(y) - \theta(x) \geq x \left(\frac{(1-\epsilon_i)B_n}{L^n} - 2\epsilon_i\right)$$

$$B_n > \frac{2\epsilon_i L_{i+1}^n}{1-\epsilon_i}$$

# 5 The active cell and analytic tail

For both orders the unique active numerical cell is $2075 \leq L < 2100$. Its order-four requirement is below $B_4$, while its order-five right-end supremum is exactly $B_5$. Because the cell is open at $L = 2100$, that supremum is not attained; the next row gives a strict positive knot margin.

Beyond $L = 10^8$, C-0053's absolute $\psi$ estimate and the standard prime-power decomposition give an absolute theta error. The certificate proves the error and its $L^n$ multiples decrease, so the prime-producing theta increment remains positive forever.

$$B_{4,\mathrm{cell}} = 60.230765700093265\ldots < B_4$$

$$B_{5,\mathrm{cell}} = \frac{252969215940000000000}{1999999999996903} = B_5$$

$$\epsilon_{\mathrm{abs}}(10^8) < 3.323858 \times 10^{-3669}$$

# 6 Pinned certificate

The certificate combines an exhaustive low prime-gap proof, exact source bridges, all corrected two-sided FKS cells, and a monotone absolute-theta tail derived from C-0053; it separately establishes global sharpness only for B4.

### Run command

```
uv run --frozen python canon/witnesses/C-0062/verify.py
```

### Pinned files

- `canon/witnesses/C-0062/verify.py`
- `canon/witnesses/C-0062/PIN.md`
- `canon/witnesses/C-0047/verify.py`
- `canon/witnesses/C-0047/PIN.md`
- `canon/witnesses/C-0053/verify.py`
- `canon/witnesses/C-0053/PIN.md`

# 7 Scope

The order-four coefficient is the least full-range value. The order-five coefficient is exact for the corrected Fiori–Kadiri–Swidinsky step-envelope splice.

# 8 Sources

- Canonical claim: `canon/claims/C-0062-axler-prime-interval-log45-improved-vector.md`
- Certificate pin: `canon/witnesses/C-0062/PIN.md`

- Existing proof: `scratch/adjacent-unconditional--axler-prime-interval-log45-improved-vector/PROOF.md`
- Independent reconstruction: `scratch/adjacent-unconditional--axler-prime-interval-log45-improved-INDEPENDENT-REVIEW.md`
- C-0047 dependency: `canon/claims/C-0047-broadbent-global-theta-log3-coefficient-0-0143406585.md`
- C-0053 dependency: `canon/claims/C-0053-fks-global-psi-amplitude-9-2202181.md`