Axler's Table 8 coefficient-1.071 row fails and has a sharp repair

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Abstract

Axler's published row $\pi(x) < x/(\log x - 1.071)$ for $x \geq 22{,}078{,}017$ fails at the prime 22,078,033. The exact same-range threshold is $a_* = \log(22{,}078{,}033) - 22{,}078{,}033/1{,}393{,}895$, so 1.07100036 is a valid strict repair. If the coefficient 1.071 is retained, the smallest valid integer start is 22,078,034.

$$\pi(x) < \frac{x}{\log x - 1.07100036} \ (x \geq 22078017), \qquad \pi(x) < \frac{x}{\log x - 1.071} \ (x \geq 22078034)$$

1 The false row and its repair

Put $x_0 = 22{,}078{,}017$, $p_* = 22{,}078{,}033$, and $n_* = 1{,}393{,}895$. At p_* the published right-hand side is smaller than $\pi(p_*)$, so the strict inequality fails inside the stated range.

The normalized value at that prime determines the exact non-strict same-range coefficient.

$$a_* = \log p_* - \frac{p_*}{n_*} = 1.071000358265763676457088876 48\ldots$$

$$\pi(p_*) - \frac{p_*}{\log p_* - 1.071} = 0.0315286257171953\ldots > 0$$

$$\pi(x) < \frac{x}{\log x - 1.07100036} \qquad (x \geq x_0)$$

2 Why only endpoints matter

Define $A(x) = \log x - x/\pi(x)$. On an interval between consecutive primes, $\pi(x)$ is constant and A decreases. At the next prime it jumps upward.

Therefore a global maximum on the published half-line can occur only at the actual starting endpoint or at a prime. This turns the finite part of the problem into an exact comparison of discrete candidates.

$$A'(x) = \frac{1}{x} - \frac{1}{n} < 0 \qquad (p_n \le x < p_{n+1})$$

$$A(p_{n+1}) - \lim_{x \uparrow p_{n+1}} A(x) = \frac{p_{n+1}}{n(n+1)} > 0$$

3 Finite maximizer and independent tail

The certificate enumerates the primes through $H = 45{,}399{,}074$ and compares 1,347,034 candidates. The unique maximum is $A(p_*) = a_*$; the published start is the unique runner-up.

Beyond H, the proof no longer relies on neighboring appendix rows. It first shows $\mathrm{li}(x) < x/(\log x - 1.070)$, then uses Buethe's strict $\pi(x) < \mathrm{li}(x)$ computation through 10^{19} and an Axler Theorem 2 reduction with coefficient 1.0343 for the remaining tail.

$$A(p_*) - A(x_0) = 0.0000006092657002968\ldots > 0$$

$$A(x) < 1.070 \qquad (x \ge H)$$

4 Sharp coefficient classification

The finite maximum and the independent tail prove that a_* is the unique global maximum of A on $x \ge x_0$. Hence the non-strict bound with coefficient a_* has equality only at p_*.

For real coefficients $a < \log x_0$, the strict upper bound on the published range holds exactly when $a > a_*$. The rounded repair 1.07100036 lies above a_* by a certified positive margin.

$$\pi(x) \le \frac{x}{\log x - a_*} \qquad (x \ge x_0)$$

$$1.07100036 - a_* = 1.7342363235\ldots \times 10^{-9} > 0$$

5 Keeping the coefficient 1.071

On the prime-free interval beginning at p_*, the crossing equation can be solved exactly with the W_{-1} branch of the Lambert W function. The principal branch is irrelevant because it gives a solution below n_*.

The published inequality fails from p_* through the crossing and holds afterward. Since the crossing lies strictly between 22,078,033 and 22,078,034, the latter is the smallest valid integer start.

$$r = -n_* W_{-1}\left(-\frac{e^{1.071}}{n_*}\right) = 22{,}078{,}033.53303818298\ldots$$

$$\pi(x) < \frac{x}{\log x - 1.071} \qquad (x \geq 22{,}078{,}034)$$

6 Pinned certificate

The pinned certificate checks the published row against exact prime counts, proves the unique finite maximizer, closes the tail independently of neighboring rows, certifies the rounded coefficient repair, and encloses the Lambert-W crossing.

Run command

```
uv run --frozen python canon/witnesses/C-0057/verify.py
```

Pinned files

- `canon/witnesses/C-0057/verify.py`
- `canon/witnesses/C-0057/PIN.md`
- `canon/witnesses/C-0057/source/repository/original-verify.py`
- `canon/witnesses/C-0057/source/repository/independent-check.py`
- `canon/witnesses/C-0057/source/repository/INDEPENDENT-REVIEW.md`

7 Scope

The repair concerns Axler's coefficient-1.071 row on $x \geq 22{,}078{,}017$ within the one-parameter denominator $x/(\log x - a)$.

8 Sources

- Canonical claim: `canon/claims/C-0057-axler-table8-1071-counterexample-sharp-repair.md`
- Certificate pin: `canon/witnesses/C-0057/PIN.md`
- Axler journal source: `canon/witnesses/C-0057/source/primary/axler-integers-24-A34.pdf`
- Axler arXiv source: `canon/witnesses/C-0057/source/primary/axler-2203.05917.pdf`

- Buethe source: `canon/witnesses/C-0057/source/primary/buthe-1511.02032.pdf`
- Independent reconstruction: `canon/witnesses/C-0057/source/repository/INDEPENDENT-REVIEW.md`