

Axler's compact prime-counting denominator coefficient improves from 70.935 to 44.053

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Abstract

For every real $x \geq 29.53$, Axler's compact upper bound for $\pi(x)$ remains valid when the final denominator coefficient 70.935 is replaced by 44.053. The proof keeps the published denominator shape and range, establishes positivity of the denominator, bridges through $\text{li}(x)$ on the low range, checks the published numerical error envelopes, and closes the asymptotic tail with a pinned absolute-error estimate.

$$\pi(x) < \frac{x}{\log x - 1 - (\log x)^{-1} - 3(\log x)^{-2} - 44.053(\log x)^{-3}} \quad (x \geq 29.53)$$

1 What is proved

Let $L = \log x$. The new denominator is larger than Axler's published denominator because the subtracted coefficient is smaller, so its reciprocal gives a strictly smaller upper bound wherever both denominators are positive.

$$\pi(x) < \frac{x}{L - 1 - L^{-1} - 3L^{-2} - 44.053L^{-3}}, \quad x \geq 29.53$$

$$70.935 - 44.053 = 26.882$$

2 Denominator and li comparison

Set $s_A(t) = 1 - t - t^2 - 3t^3 - At^4$ with $A = 44.053$. After dividing by x/L , the gap between the proposed reciprocal denominator and $\text{li}(x)$ is expressed through $s_A(1/L)$ and the exponential integral.

The certificate proves both the new and published denominators are positive at the lower endpoint and remain positive thereafter. It then covers $3.385 \leq L \leq 10$ by 16384 directed rational cells.

$$B_{\text{exact}}(L) = \frac{1}{s_A(1/L)} - Le^{-L} \text{Ei}(L)$$

$$\min B_{\text{exact}}(L) > 1.0957 \times 10^{-3} \quad (3.385 \leq L \leq 10)$$

3 Rational majorants and the finite bridge

For $L \geq 10$, a short rational majorant for $e^{-L} \text{Ei}(L)$ has positive initial gap and positive differential forcing. Exact multiplication shows that the proposed reciprocal denominator dominates this majorant.

This proves that the proposed right-hand side exceeds $\text{li}(x)$. Buethé's strict computation $\pi(x) < \text{li}(x)$ through 10^{19} then supplies the prime-counting inequality over the corresponding range.

$$u(L) = L^{-1} + L^{-2} + 2L^{-3} + 6L^{-4} + 55L^{-5}$$

$$1 - s_A(t)(1 + t + 2t^2 + 6t^3 + 55t^4) = \frac{t^4}{1000} (53 + 111053t + 161106t^2 + 429318t^3 + 2422915t^4) > 0$$

4 Published numerical envelopes

A higher-order rational majorant handles $L \geq 40$. Fiori–Kadiri–Swidinsky Table 7 covers the initial high range, and the independently source-checked 164-row Table 4 transcript covers the numerical intervals through $L = 3000$.

On each interval the proof computes the exact coefficient required by that row. The unique largest requirement occurs on $2100 \leq L \leq 2200$ and is strictly below 44.053.

$$A_{\text{req}} = \frac{2709976113860085547103395243942841453423728245400}{61517068085059134496927148872832330483236342381} = 44.05242639511077 \dots$$

$$44.053 - A_{\text{req}} = 0.000573604889227 \dots > 0$$

5 Asymptotic closure and certificate

For $L \geq 3000$, exact polynomial arithmetic gives a simple positive lower bound for the reciprocal-denominator gap. A pinned global absolute estimate for $\pi(x) - \text{Li}(x)$ closes the tail, with the source identity $\text{Li}(x) = \text{li}(x) - \text{li}(2)$ retained.

At the handoff the certified margin exceeds 1.8130×10^{-13} and the comparison improves afterward. The pinned certificate checks the imported tail dependency before checking every local gate.

$$\text{tail margin at } L = 3000 > 1.8130 \times 10^{-13}$$

6 Pinned certificate

The pinned certificate checks denominator positivity, the 16384-cell low li cover, two rational Ei majorants, exact residual polynomials, Table 7, all 164 source-checked Table 4 rows, and the asymptotic handoff.

Run command

```
uv run --frozen python canon/witnesses/C-0055/verify.py
```

Pinned files

- canon/witnesses/C-0055/verify.py
- canon/witnesses/C-0055/PIN.md
- canon/witnesses/C-0055/results.json
- canon/witnesses/C-0055/source/dependency/c0050/results.json

7 Scope

The certified coefficient 44.053 preserves Axler’s denominator shape and applies for every real $x \geq 29.53$.

8 Sources

- Canonical claim: canon/claims/C-0055-axler-compact-pi-denominator-44-053.md
- Certificate pin: canon/witnesses/C-0055/PIN.md
- Accepted certificate data: canon/witnesses/C-0055/results.json
- Axler source: canon/witnesses/C-0055/source/primary/axler-integers-24-A34.pdf
- Buethe source: canon/witnesses/C-0055/source/primary/buthe-1511.02032.pdf
- Fiori–Kadiri–Swidinsky source: canon/witnesses/C-0055/source/primary/fiori-kadiri-swidinsky-220612557v2.pdf