Broadbent et al.'s global theta coefficient improves from 0.024334 to 0.0143406585

Kenan

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Abstract

A global upper bound for the Chebyshev function $\theta(x) = \sum_{p \le x} \log p$ is obtained by splicing a verified small-x sign result, 193 published numerical step envelopes, and a decreasing analytic tail. The exact maximum of the normalized rounded step envelopes is 0.0143406585.

$$\theta(x) < x\left(1 + \frac{0.0143406585}{\log^3 x}\right) \qquad (x > 1)$$

1 Statement and normalization

The claim controls only the upper signed error of $\theta(x)$. Writing the relative error as $E_\theta(x)$ shows that the coefficient is a uniform upper bound for $E_\theta(x)\log^3 x$.

The resulting coefficient replaces 0.024334 in the same global inequality and on the same range.

$$E_\theta(x) = \frac{\theta(x) - x}{x}$$

$$E_\theta(x)\log^3 x < 0.0143406585 \qquad (x > 1)$$

2 Small values and overlap

A theorem of Buethe gives $\theta(x) < x$ throughout $1 \le x \le 10^{19}$. This is stronger than the target upper bound because the added $\log^{-3} x$ term is positive.

The numerical table regime begins at $\log x = 37$, while $\log(10^{19})$ lies between 43 and 44. The two inputs therefore overlap rather than leaving a gap.

$$x - \theta(x) > 0.05\sqrt{x} \qquad (1 \le x \le 10^{19})$$

$$43 < 19 \log 10 < 44$$

3 Finite step envelopes

Each Fiori–Kadiri–Swidinsky row (L_i, ϵ_i) gives $E_\theta(x) \le \epsilon_i$ for all $x \ge e^{L_i}$. On an interval between table knots, every earlier row remains available, so the best inherited value is the running minimum ϵ_i^*.

Since L^3 increases across the interval, its limiting right-end product controls that interval. Exact rational comparison of all 193 rows finds one maximal interval.

$$\epsilon_i^* = \min_{j \le i} \epsilon_j$$

$$E_\theta(x) L^3 \le \epsilon_i^* L^3 < \epsilon_i^* L_{i+1}^3$$

$$(1.5485 \times 10^{-12})\, 2100^3 = 0.0143406585$$

1. Form the running minimum of the admissible table rows.
2. Evaluate each interval's right-end majorant exactly.
3. Check each knot using the newly available row.

4 Typo repair and strictness

The first two displayed theta entries in Table 2 have an exponent inconsistency. Proposition 17 applied to the same row yields approximately 1.9536449×10^{-8}, so the outward rounded admissible value is 1.9537×10^{-8} rather than the printed 10^{-9} exponent.

This repaired row does not determine the maximum. The maximum comes from $2075 \le L < 2100$ and is an unattained right-end supremum. At $L = 2100$, a stronger row becomes available and gives 0.0122671206, so every actual knot value is strictly below the coefficient.

$$\epsilon_{\theta,\mathrm{num}}(10^{19}) = 1.953644927822\ldots \times 10^{-8} < 1.9537 \times 10^{-8}$$

$$(1.3246 \times 10^{-12})\, 2100^3 = 0.0122671206 < 0.0143406585$$

5 Analytic tail

For $L = \log x \geq 3100$, the Fiori–Kadiri–Swidinsky analytic envelope replaces the tables. After multiplying by L^3, logarithmic differentiation shows the normalized tail decreases once $L > 81R/4$.

With $R = 5.5666305$, this decrease begins far before the handoff. Directed evaluation at $L = 3100$ gives a normalized tail below 0.000151, well below the finite-envelope coefficient.

$$E_\theta(x) \leq 121.0961 \left(\frac{L}{R}\right)^{3/2} \exp\left(-2\sqrt{\frac{L}{R}}\right)$$

$$\frac{h'(L)}{h(L)} = \frac{9}{2L} - \frac{1}{\sqrt{RL}} < 0 \qquad \left(L > \frac{81R}{4}\right)$$

$$h(3100) < 0.000151 < 0.0143406585$$

6 Pinned certificate

The pinned certificate checks the repaired FKS source row, all 193 exact rational step envelopes, the unique interval supremum at [2075,2100), strict knot values, overlap with the small-x range, and the decreasing Corollary 14 tail.

Run command

```
uv run --frozen python canon/witnesses/C-0047/verify.py
```

Pinned files

- `canon/witnesses/C-0047/verify.py`
- `canon/witnesses/C-0047/PIN.md`

7 Scope

The coefficient is exact for the stated splice of published rounded-up upper step envelopes and controls the upper signed error.

8 Sources

- Canonical claim: `canon/claims/C-0047-broadbent-global-theta-log3-coefficient-0-0143406585.md`

- Pinned certificate note: `canon/witnesses/C-0047/PIN.md`
- Certificate source: `experiments/fks_global_theta_log3_coefficient/verify.py`