The sharp Rosser–Schoenfeld weighted-prime-sum upper coefficient

Kenan

2026-07-19

Abstract

For $S(x) = \sum_{p \le x} (\log p)/p$, the exact upper coefficient in the Rosser–Schoenfeld formula on $x \ge 319$ is attained uniquely at $x = 467$. A separate splice of finite-range positivity with a global analytic estimate gives a lower coefficient $C_{\text{low}} < 1/221$ for every real $x > 1$.

$$C_{\text{up}} = 0.492206382859887\ldots, \qquad C_{\text{low}} = 0.004513569432713\ldots$$

1 Upper and lower statements

The constant B_3 centers the weighted prime sum around its main term $\log x$. The claim treats the two signed errors separately: an exact same-range upper optimum and a non-optimal global lower coefficient.

$$S(x) = \sum_{p \le x} \frac{\log p}{p}, \qquad B_3 = \gamma + \sum_{p} \frac{\log p}{p(p-1)}$$

$$S(x) \le \log x - B_3 + \frac{C_{\text{up}}}{\log x} \qquad (x \ge 319)$$

$$S(x) > \log x - B_3 - \frac{C_{\text{low}}}{\log x} \qquad (x > 1)$$

2 Upper normalization

Multiplying the upper error by $\log x$ converts the coefficient problem into maximizing a function that is smooth between prime jumps. On such an interval the weighted prime sum is constant.

The derivative is negative at the composite endpoint 319 and after every later checked prime. Thus the only candidates are $x = 319$ and the values immediately after prime jumps.

$$F(x) = \log x \,(S(x) - \log x + B_3)$$

$$F'(x) = \frac{S(x) + B_3 - 2\log x}{x}$$

3 Unique upper maximum

The finite comparison identifies $x = 467$ as the unique maximizer. It lies only about 6.85×10^{-5} above the endpoint value at 319, so the endpoint comparison is one of the load-bearing numerical checks.

Beyond 912560, Dusart's estimate bounds the normalized upper error by $0.3/\log x$, which is below 0.022 at the handoff and then decreases.

$$C_{\mathrm{up}} = F(467) = 0.4922063828598873676216976896219548566\ldots$$

$$|S(x) - \log x + B_3| \le \frac{0.3}{\log^2 x} \qquad (x \ge 912560)$$

$$F(467) - F(319) > 6.8 \times 10^{-5}$$

1. Prove decrease on each prime-free interval.
2. Compare the endpoint and every finite prime jump.
3. Separate the candidate from the analytic tail.

4 Global lower splice

Rosser and Schoenfeld proved $S(x) > \log x - B_3$ through 10^8, so the desired negative lower allowance is automatic there. Axler's global estimate supplies the tail.

After factoring out $1/\log x$, the remaining coefficient function is decreasing. Its endpoint value at 10^8 is C_{low}, which gives a strict bound beyond the splice.

$$S(x) - \log x + B_3 \ge -\frac{3}{40\log^2 x} - \frac{3}{20\log^3 x}$$

$$g(L) = \frac{3}{40L} + \frac{3}{20L^2}, \qquad C_{\mathrm{low}} = g(\log 10^8)$$

$$C_{\mathrm{low}} = 0.004513569432713956030838616810073784\ldots < \frac{1}{221}$$

5 Certificate role

The certificate encloses B_3 through a Mobius-inverted zeta-derivative series with an explicit omitted tail. It enumerates the primes through 912560, certifies the inter-prime derivative signs, proves the unique finite maximum at 467, and checks the Dusart tail margin.

It also evaluates the lower splice coefficient and proves $C_{\text{low}} < 1/221$. The cited source theorems, their ranges, and the symbolic monotonicity of the lower coefficient function remain analytic inputs to the written proof.

$$B_3 = \gamma + \sum_{m=2}^{\infty} \mu(m) \frac{\zeta'(m)}{\zeta(m)}$$

$$C_{\text{up}} < 0.492207 < \frac{1}{2}, \qquad C_{\text{low}} < \frac{1}{221}$$

6 Pinned certificate

The pinned certificate rigorously encloses B3, checks every finite prime-jump and derivative comparison needed for the unique upper maximum at x=467, separates the Dusart tail, and certifies the lower splice constant and its 1/221 rounding.

Run command

```
uv run --frozen python canon/witnesses/C-0045/verify.py
```

Pinned files

- `canon/witnesses/C-0045/verify.py`
- `canon/witnesses/C-0045/PIN.md`

7 Scope

The upper coefficient is sharp on $x \geq 319$ for the stated $1/\log x$ formula; the lower coefficient is a certified global value for the companion formula.

8 Sources

- Canonical claim: `canon/claims/C-0045-sharp-rosser-schoenfeld-weighted-prime-sum-coefficients.md`
- Pinned certificate note: `canon/witnesses/C-0045/PIN.md`

- Certificate source: `experiments/weighted_prime_sum_rs_coefficients/verify.py`