The sharp full-range Rosser–Schoenfeld product and totient coefficients

Kenan

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Abstract

Two classical full-range inequalities have exact coefficients within their original formula shapes. For the Mertens prime product on $x \geq 286$, the normalized maximum occurs uniquely at $x = 286$. For $n/\varphi(n)$ on $n \geq 3$, the normalized maximum occurs uniquely at the exceptional primorial $N_9 = 223092870$.

$$C_{\text{prod}} = 0.482761856265219\ldots, \qquad C_\varphi = 2.506369279696429\ldots$$

1 The two statements

Let $P(x) = \prod_{p \leq x} p/(p-1)$. The first result determines the least coefficient in the Rosser–Schoenfeld product bound on the unchanged range $x \geq 286$.

The second result determines the least coefficient in their global totient formula for integers $n \geq 3$.

$$P(x) \leq e^\gamma \log x \left(1 + \frac{C_{\text{prod}}}{\log^2 x}\right) \qquad (x \geq 286)$$

$$\frac{n}{\varphi(n)} \leq e^\gamma \log\log n + \frac{C_\varphi}{\log\log n} \qquad (n \geq 3)$$

2 Product normalization

The product inequality is normalized so that its coefficient is the maximum of a stepwise-smooth function. Since no prime lies between 283 and 293, the candidate at $x = 286$ uses the Euler product through 283.

On an interval with no prime, the product is constant. The derivative test shows that the normalized error decreases between successive prime jumps, reducing the finite problem to the endpoint and values immediately after primes.

$$E(x) = \log x \left(e^{-\gamma} P(x) - \log x\right)$$

$$C_{\text{prod}} = E(286) = \log 286 \left(e^{-\gamma} \prod_{p \le 283} \frac{p}{p-1} - \log 286 \right)$$

$$E'(x) = \frac{e^{-\gamma} P(x) - 2\log x}{x} < 0$$

3 Product maximum and tail

Every later prime jump through the Dusart handoff is strictly below $E(286)$. The nearest competitor is $p = 293$, separated by more than 0.0236028.

Dusart's product estimate gives a normalized tail at most $1/(5\log x)$. At the handoff this is below 0.014, far below the candidate, so $x = 286$ is the unique global maximizer on the stated range.

$$C_{\text{prod}} = 0.4827618562652196906698586265933631\ldots$$

$$P(x) \le e^{\gamma} \log x \left(1 + \frac{1}{5\log^3 x}\right) \qquad (x \ge 2\,278\,382)$$

4 Totient exceptional value

Rosser and Schoenfeld proved the stronger coefficient 5/2 for every $n \ge 3$ except one integer. That exception is the product of the first nine primes.

Evaluating the normalized error at this primorial gives a value strictly between 5/2 and the published exception-covering decimal 2.50637. Since every other integer has normalized error below 5/2, the exceptional primorial is the unique global maximizer.

$$N_9 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 = 223092870$$

$$C_\varphi = \log\log N_9 \left(\frac{N_9}{\varphi(N_9)} - e^{\gamma} \log\log N_9 \right)$$

$$C_\varphi = 2.5063692796964296773138209307904675 88\ldots$$

5 Certificate role

For the product result, the pinned certificate enumerates all 168064 primes through the handoff, checks negative derivative on every prime-free interval, compares every later jump, and verifies the analytic tail margin.

For the totient result, it constructs the exceptional primorial exactly, evaluates its Euler ratio and logarithms with directed Arb arithmetic, and certifies both the sharp value and the strict decimal rounding $C_\varphi < 2.50636928 < 2.50637$. The conclusion for all other integers uses the cited 5/2 theorem.

$$C_{\mathrm{prod}} < 0.482762 < \frac{1}{2}$$

$$\frac{5}{2} < C_\varphi < 2.50636928 < 2.50637$$

6 Pinned certificate

The pinned Arb certificate proves the product endpoint is the unique finite prime-jump maximum and lies above the Dusart tail, then evaluates the unique Rosser–Schoenfeld totient exception and its strict roundings. The all-other-n totient step is supplied by the cited theorem.

Run command

```
uv run --frozen python canon/witnesses/C-0042/verify.py
```

Pinned files

- `canon/witnesses/C-0042/verify.py`
- `canon/witnesses/C-0042/PIN.md`

7 Scope

Both coefficients are exact for the displayed Rosser–Schoenfeld product and totient formulas on their original ranges.

8 Sources

- Canonical claim: `canon/claims/C-0042-sharp-rosser-schoenfeld-product-totient-coefficients.md`

- Pinned certificate note: `canon/witnesses/C-0042/PIN.md`
- Certificate source: `experiments/mertens_product_rs_sharp_coefficient/verify.py`