The sharp Rosser–Schoenfeld reciprocal-prime coefficient is 0.487203269718814...

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Abstract

Let $A_1(x) = \sum_{p\le x} p^{-1} - \log\log x - B$, where B is the Meissel–Mertens constant. On the fixed range $x \ge 286$, the largest value of $A_1(x)\log^2 x$ occurs uniquely at $x = 286$. Its value is $C_* = 0.487203269718814568\ldots$, giving the exact coefficient for the Rosser–Schoenfeld formula shape on that range.

$$A_1(x) \le \frac{C_*}{\log^2 x} \ (x \ge 286), \qquad C_* = 0.487203269718814\ldots$$

1 Statement and normalization

The reciprocal-prime Mertens error compares the partial sum of $1/p$ with its main term $\log\log x + B$. Multiplying the desired inequality by $\log^2 x$ turns the coefficient question into a maximization problem for a normalized error.

Because there is no prime between 283 and 293, the value at the left endpoint uses the reciprocal-prime sum through 283. The candidate coefficient is therefore an exact expression in that finite sum and the constant B.

$$A_1(x) = \sum_{p\le x} \frac{1}{p} - \log\log x - B$$

$$C_* = A_1(286)\log^2 286 = \left(\sum_{p\le 283} \frac{1}{p} - \log\log 286 - B\right)\log^2 286$$

2 Classical context

Rosser and Schoenfeld proved the same upper-bound shape with coefficient $1/2$ for every real $x \ge 286$. The present statement keeps both that formula and that starting point, replacing only the coefficient.

A convenient rounded consequence is $C_* < 121801/250000 = 0.487204 < 1/2$.

$$A_1(x) < \frac{1}{2\log^2 x} \qquad (x \geq 286)$$

$$A_1(x) < \frac{121801}{250000\log^2 x} \qquad (x \geq 286)$$

3 Reduction to prime jumps

Fix $C = C_*$ and subtract the comparison term. Between consecutive primes, the reciprocal-prime sum is constant, so the resulting function is differentiable and strictly decreasing on the entire range.

At a prime, A_1 jumps upward by $1/p$. It is therefore enough to compare the endpoint value with the normalized error immediately after every later prime jump. The value at 286 is zero after subtraction, while all later checked jumps are negative.

$$D_C(x) = A_1(x) - \frac{C}{\log^2 x}$$

$$D'_C(x) = \frac{-\log^2 x + 2C}{x\log^3 x} < 0 \qquad (x \geq 286)$$

1. Establish strict decrease on each prime-free interval.
2. Compare C_* with every normalized prime-jump value from $p = 293$ to the analytic handoff.
3. Use a published tail estimate beyond the handoff.

4 Closing the infinite tail

The finite comparison ends at the threshold in Dusart's reciprocal-prime estimate. That estimate is stronger by one power of $\log x$ than the target shape, so after normalization its right-hand side decreases with x.

At the handoff the normalized tail bound is already more than 0.473 below C_*. This closes every remaining real x without further prime enumeration.

$$|A_1(x)| \leq \frac{1}{5\log^3 x} \qquad (x \geq 2\,278\,383)$$

$$A_1(x)\log^2 x \leq \frac{1}{5\log x} < C_*$$

5 Pinned certificate

Directed 256-bit Arb arithmetic encloses the Meissel–Mertens constant, certifies the unique finite maximum at x=286 over every later prime jump through 2,278,383, and verifies that Dusart's normalized tail remains strictly below the candidate.

Run command

```
uv run --frozen python canon/witnesses/C-0039/verify.py
```

Pinned files

- `canon/witnesses/C-0039/verify.py`
- `canon/witnesses/C-0039/PIN.md`

6 Scope

The coefficient is exact for the upper reciprocal-prime error on the fixed range $x \geq 286$ and the stated $C/\log^2 x$ formula.

7 Sources

- Canonical claim: `canon/claims/C-0039-sharp-rosser-schoenfeld-mertens-coefficient.md`
- Pinned certificate note: `canon/witnesses/C-0039/PIN.md`
- Certificate source: `experiments/mertens_rs_sharp_coefficient/verify.py`