Pi/6 is the sharp zero-sector threshold for centered factorial D3 positivity

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Abstract

For a finite conjugation-stable multiset of at least two nonzero reciprocal zeros, confinement to the closed sector $|\arg\alpha| \le \pi/6$ forces positivity of the cubic power-sum invariant governing the centered factorial 3×3 derivative determinant. The proof rewrites that invariant as a positive combination of elementary symmetric sums of α^3 and α^2. Both powered multisets lie in the closed right half-plane at exactly this angle. A one-real-root plus conjugate-pair family becomes negative in every wider sector, and a small genus-zero tail turns it into a transcendental order-zero counterexample. The entire-function consequence follows by differentiating, preserving the zero sector, and applying the exact Newton determinant identity.

$$A = \overline{A},\ |A| \ge 2,\ |\arg\alpha_\nu| \le \frac{\pi}{6} \Longrightarrow E(A) > 0$$

1 What is proved

Let $A = (\alpha_\nu)$ be a finite conjugation-stable multiset of at least two nonzero complex numbers, and put $q_k = \sum_\nu \alpha_\nu^k$. If every element lies in the closed sector of half-angle $\pi/6$, then the invariant $E(A)$ is strictly positive.

The angle is optimal in the universal statement: every sector with half-angle greater than $\pi/6$ contains both a finite conjugation-stable counterexample and a real transcendental entire counterexample of order zero with positive Maclaurin coefficients.

$$E(A) = q_2^3 - 3q_2q_4 + 2q_3^2$$

$$|\arg\alpha_\nu| \le \frac{\pi}{6} \quad \Longrightarrow \quad E(A) > 0$$

2 The exact algebraic identity

Newton identities reorganize the power-sum expression without estimating its individual monomials. The decisive formula groups roots before taking signs.

This grouping is stronger than requiring each raw monomial to have positive real part. It is what enlarges the earlier sufficient half-angle to the closed value $\pi/6$.

$$E(A) = 4e_2(\alpha^3) + 6e_3(\alpha^2)$$

$$e_2(\alpha^3) = \sum_{\mu<\nu} \alpha_\mu^3 \alpha_\nu^3, \qquad e_3(\alpha^2) = \sum_{\lambda<\mu<\nu} \alpha_\lambda^2 \alpha_\mu^2 \alpha_\nu^2$$

3 Why the boundary angle works

A conjugation-stable multiset in the closed right half-plane has nonnegative elementary symmetric sums. A positive real singleton contributes $1 + rt$, while a nonreal conjugate pair contributes a quadratic with nonnegative coefficients.

At $|\arg\alpha| \le \pi/6$, both α^3 and α^2 lie in the closed right half-plane. Hence both terms in the identity are nonnegative. The first is strictly positive for every conjugation-stable nonzero multiset with at least two elements: a conjugate pair contributes $|\alpha|^6$, two positive real singletons have positive product, and cross-block terms are nonnegative.

$$(1 + \beta t)(1 + \overline{\beta} t) = 1 + 2\,\mathrm{Re}(\beta)t + |\beta|^2 t^2$$

$$e_2(\alpha^3) > 0, \qquad e_3(\alpha^2) \ge 0$$

4 Sharpness beyond pi over six

Choose $\pi/6 < \theta < \min(\epsilon, \pi/2)$ and use one positive real reciprocal zero together with a conjugate pair. The exact invariant has a negative cubic leading term because $\cos(3\theta) < 0$, so it is negative for sufficiently large t.

The corresponding cubic polynomial has positive coefficients. Multiplying it by a sufficiently small product $\prod_{m\ge1}(1 + \delta 2^{-m} z)$ preserves the negative invariant by continuity, while producing a real transcendental entire function of order zero with positive coefficients in every degree.

$$A_t = \{t, e^{i\theta}, e^{-i\theta}\}, \qquad E(A_t) = 4 + 6t^2 + 8t^3\cos(3\theta)$$

$$E(A_{10})\big|_{\theta=\pi/5} = 2604 - 2000\sqrt{5} < 0$$

5 Entire-function transport

Let F be real, transcendental, of order below one, with all derivatives at zero positive and all zeros in $|\arg(-z)| \leq \pi/6$. For a fixed $n \geq 2$, set $G = F^{(n-2)}$. Convexity of the sector, genus-zero partial products, Gauss-Lucas, Hurwitz, and local uniform convergence keep the zeros of G in the same sector.

Hadamard factorization gives an absolutely convergent genus-zero product. Transcendence ensures that G has infinitely many zeros, so the strict elementary-symmetric contribution does not disappear. The exact Newton identity then converts $E(A) > 0$ into the centered factorial derivative determinant.

$$G(z) = G(0) \prod_{\nu} (1 + \alpha_\nu z), \qquad \sum_{\nu} |\alpha_\nu| < \infty$$

$$\det \begin{pmatrix} G''(0) & G'''(0) & G^{(4)}(0) \\ G'(0) & G''(0) & G'''(0) \\ G(0) & G'(0) & G''(0) \end{pmatrix} = 2G(0)^3 E(A) > 0$$

$$\det[F^{(n+j-i)}(0)]_{i,j=0}^{2} > 0 \qquad (n \geq 2)$$

6 Certificate role

The pinned verifier checks the determinant-to-power-sum reduction, the universal elementary-symmetric identity, direct finite-root collision patterns, the conjugate-pair right-half-plane factor, the sharp three-root family, the exact algebraic witness, and an explicit negative order-zero transcendental witness.

The all-sector quantifier and the passage to derivatives of an arbitrary order-below-one entire function use the written Gauss-Lucas, Hurwitz, Hadamard, convergence, and strictness arguments. The finite certificate supports those steps but does not replace them.

7 Pinned certificate

The verifier certifies the symbolic Newton identities and the explicit finite and transcendental sharpness witnesses. The universal sector proof and the entire-function transport are completed by the written conjugate-pair, Gauss-Lucas, Hurwitz, Hadamard, and convergence arguments.

Run command

```
uv run --frozen python canon/witnesses/C-0038/verify.py
```

Pinned files

- `canon/witnesses/C-0038/PIN.md`
- `canon/witnesses/C-0038/verify.py`

8 Scope

The sharp $\pi/6$ threshold applies to centered factorial order-three determinants for the stated conjugation-stable reciprocal-zero multisets and their order-below-one entire-function transport.

9 Sources

- Canonical claim: `canon/claims/C-0038-pi-over-6-is-sharp-factorial-d3-sector-threshold.md`
- Witness pin: `canon/witnesses/C-0038/PIN.md`
- Proof result: `scratch/adjacent-unconditional--factorial-d3-sharp-sector-threshold/RESULT.md`
- Prior-art boundary: `literature/2026-07-19-factorial-d3-sharp-sector-threshold-search.md`
- Pinned proof source: `scratch/adjacent-unconditional--factorial-d3-sharp-sector-threshold/verify.py`