A certified Polya-frequency order of 9,425,328,785,004

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Abstract

The evenness of $\xi(1/2+w)$ makes $\xi_1(z)=\xi(1/2+\sqrt{z})$ a real entire function of order $1/2$. A verified zero height T constrains every zero of ξ_1 to lie outside a large sector about the positive real axis. Schoenberg's converse sector theorem converts that zero-free sector into finite Polya-frequency order. Platt and Trudgian's published height $T=3,000,175,332,800$ gives $\xi_1\in PF_{9,425,328,785,004}$ through the stated worst-case sector bound.

$$\xi\big(\tfrac{1}{2}+\sqrt{z}\big)\in PF_{9425328785004}$$

1 What is proved

Set $\xi_1(z)=\xi(1/2+\sqrt{z})$. The theorem proves that its Maclaurin coefficient sequence is Polya-frequency of every order up to $9,425,328,785,004$.

The reusable statement is parameterized: if all nontrivial zeta zeros through height T lie on the critical line, then $\xi_1\in PF_m$ whenever $\pi/(m+1)\geq 2\arctan(1/(2T))$.

$$RH(T)\quad\Longrightarrow\quad \xi_1\in PF_m\ \text{ for }\ \frac{\pi}{m+1}\geq 2\arctan\!\left(\frac{1}{2T}\right)$$

2 Moving zeta zeros into the z-plane

Write a nontrivial zeta zero as $\rho=1/2+u+i\gamma$. Under the square map used to define ξ_1, it becomes $z_\rho=(u+i\gamma)^2$.

Zeros with $|\gamma|\leq T$ lie on the critical line by hypothesis, so they map to the negative real axis. For the remaining zeros, the critical strip gives $|u|<1/2$, and their angular defect from the negative axis is bounded by $2\arctan(1/(2T))$.

$$z_\rho=(\rho-\tfrac{1}{2})^2=(u+i\gamma)^2$$

$$\pi-|\arg z_\rho|=2\arctan\!\left(\frac{|u|}{|\gamma|}\right)<2\arctan\!\left(\frac{1}{2T}\right)$$

3 The sector-to-PF transfer

Because ξ_1 has order $1/2$, its zeros admit a genus-zero product with summable reciprocal moduli. Truncating by complete negative-real factors and conjugate pairs produces real polynomials with positive constant term and the same zero-free sector.

Schoenberg's converse theorem says that a real polynomial with positive constant term and no zero in $|\arg z| < \pi m/(m+1)$ belongs to PF_m. Local uniform convergence of the real partial products gives coefficientwise convergence, and every fixed Toeplitz minor remains nonnegative in the limit.

$$P_N(z) \neq 0 \quad \text{for} \quad |\arg z| < \frac{\pi m}{m+1} \quad \Longrightarrow \quad P_N \in PF_m$$

4 The certified endpoint

Platt and Trudgian's published finite-height theorem supplies $T = 3000175332800$. The pinned interval calculation places $\pi/[2\arctan(1/(2T))]$ strictly between $9,425,328,785,005$ and $9,425,328,785,006$.

The condition is on $m+1$, so the largest integer order certified by this particular worst-case sector inequality is $m = 9,425,328,785,004$.

$$9425328785005 < \frac{\pi}{2\arctan(1/(2T))} < 9425328785006$$

$$\xi_1 \in PF_{9425328785004}$$

5 What the certificate checks

The witness pins Schoenberg's sector theorem through Katkova's published application and checks the endpoint arithmetic using Platt and Trudgian's published height. It also checks the square-map multiplicity argument, the real conjugate-orbit truncation, and closure of each fixed Toeplitz minor under the entire-function limit.

It also records two repairs to the earlier printed application. A displayed sector of $43\pi/44$ meets the PF_{43} threshold, not the PF_{44} threshold, and arbitrary zero truncations need not be real. Grouping complete conjugate orbits repairs the latter issue.

$$\frac{43\pi}{44} = \frac{\pi \cdot 43}{43+1}$$

6 Pinned certificate

The certificate pins the converse sector theorem and the finite-height dependency, checks the zero-square angular geometry, real partial-product construction, limit closure, corrected order indexing,

and the interval arithmetic selecting the stated finite order.

Run command

```
uv run --frozen python canon/witnesses/C-0026/verify.py
```

Pinned files

- `canon/witnesses/C-0026/PIN.md`
- `canon/witnesses/C-0026/verify.py`
- `canon/witnesses/C-0026/source/schoenberg-1955/publisher-page.html`
- `canon/witnesses/C-0026/source/katkova-2007/arxiv-v1.tex`
- `canon/witnesses/C-0026/source/katkova-2007/journal.pdf`
- `canon/claims/C-0026-verified-zeta-height-implies-xi1-pf9425328785007.md`

7 Scope

The stated Polya-frequency order is the largest integer certified by the displayed worst-case sector inequality at Platt and Trudgian's published height $3,000,175,332,800$.

8 Sources

- Canonical claim: `canon/claims/C-0026-verified-zeta-height-implies-xi1-pf9425328785007.md`
- Witness pin: `canon/witnesses/C-0026/PIN.md`
- Katkova source text: `canon/witnesses/C-0026/source/katkova-2007/arxiv-v1.tex`
- Finite-order literature audit: `literature/2026-07-18-xi1-pf-verified-height-search.md`
- Published finite-height theorem: D. J. Platt and T. S. Trudgian, "The Riemann hypothesis is true up to $3 \cdot 10^{12}$," Bulletin of the London Mathematical Society 53 (2021), 792–797. `https://doi.org/10.1112/blms.12460`