All-order positive kernels for curves over finite fields

Kenan

2026-07-18

Abstract

For a smooth projective geometrically connected curve $C/\mathbf{F}_q$, the numerator of its zeta function can be centered into an even entire function X_C. The Rosati involution on the Jacobian forces every Frobenius root to have modulus $\sqrt{q}$. This puts the zeros of X_C on explicit imaginary frequency towers, and the logarithmic derivative becomes a Stieltjes sum whose negative divided difference is a Gram kernel. A positive diagonal congruence then proves that the descended kernel Q_C is positive semidefinite at every matrix order and every positive node.

$$Q_C(\lambda_i, \lambda_j) \succeq 0 \quad (\lambda_i > 0, \text{ all matrix orders})$$

1 What is proved

Let P_C be the numerator of the zeta function of $C/\mathbf{F}_q$, and center it by $X_C(z) = e^{g(\log q)z} P_C(q^{-1/2} e^{-(\log q)z})$. Write $L_C = X_C'/X_C$.

For positive nodes, the theorem concerns the confluent kernel $Q_C(\lambda, \mu) = 4[\mu L_C(\lambda) - \lambda L_C(\mu)]/(\mu^2 - \lambda^2)$. Its diagonal value is defined by continuity. Every finite matrix formed from this kernel is positive semidefinite.

$$Q_C(\lambda_i, \lambda_j) \succeq 0 \qquad (\lambda_i > 0)$$

2 The polarization input

Let $J = \mathrm{Jac}(C)$ and let π be its q-power Frobenius endomorphism. A polarization gives the positive Rosati involution $\dagger$, and the pinned source identity is $\pi^\dagger \pi = [q]_J$.

Rosati positivity also shows directly that $\mathbf{Q}[\pi]$ is a product of fields stable under $\dagger$. On each complex factor, the involution acts as complex conjugation. Therefore every characteristic root α of Frobenius satisfies $|\alpha|^2 = q$.

$$\pi^\dagger \pi = [q]_J, \qquad |\alpha| = \sqrt{q}$$

3 From point counts to an even function

The numerator identity is derived from the point-count formula over every finite extension, rather than imported from a source passage with a polynomial-convention mismatch. Formal logarithmic summation identifies $P_C(u) = \det(1 - u\pi \mid V_\ell J)$.

Integrality and purity pair each root with q/α, including multiplicity. The resulting functional equation makes X_C even and periodic. The fixed normalized roots $+1$ and -1 have even multiplicity, so the exceptional zero frequencies can be grouped without losing their multiplicities.

$$N_n = 1 + q^n - \sum_j \alpha_j^n$$

$$P_C(u) = q^g u^{2g} P_C\left(\frac{1}{qu}\right)$$

4 The Gram-kernel route

The paired frequency product has no hidden exponential term because X_C is even. If m_0 is the multiplicity at zero and m_γ is the multiplicity at $i\gamma$, logarithmic differentiation gives a Stieltjes expansion.

With $H_C(x) = X_C'(\sqrt{x})/(\sqrt{x}\, X_C(\sqrt{x}))$, its negative divided difference is a sum of rank-one positive kernels. Setting $x_i = \lambda_i^2$ then gives $Q = DMD$ with a positive diagonal matrix D.

$$H_C(x) = \frac{m_0}{x} + 2\sum_{\gamma>0} \frac{m_\gamma}{x + \gamma^2}$$

$$M_C(x, y) = \frac{m_0}{xy} + 2\sum_{\gamma>0} \frac{m_\gamma}{(x + \gamma^2)(y + \gamma^2)}$$

$$Q = DMD, \qquad D = \operatorname{diag}(2\lambda_i)$$

5 What the certificate checks

The pinned certificate checks the Milne source hashes and theorem anchors, reconstructs the power-sum derivation of the zeta numerator, and verifies the reciprocal-pair and exceptional-multiplicity bookkeeping. It also checks the confluent diagonal and the exact diagonal congruence.

As exact fixtures, it reconstructs five curve numerators from finite-field point counts. The curve $y^2 = x^5 - x$ over $\mathbf{F}_5$ has numerator $(1 - 5u^2)^2$ and exercises a zero at the origin, both self-reciprocal angles, and repeated frequencies. Arb enclosures certify all principal minors of direct 3×3 test matrices at positive nodes including $1/4$.

$$P_C(u) = (1 - 5u^2)^2, \qquad X_C(z) = 4\sinh^2((\log 5)z)$$

6 Pinned certificate

The certificate pins the polarization and point-count sources, checks the exact Frobenius-to-canonical-product derivation, verifies the Gram decomposition and confluent kernel identity, and exercises the exceptional multiplicities on exact finite-field curves.

Run command

```
uv run --frozen python canon/witnesses/C-0016/verify.py
```

Pinned files

- `canon/witnesses/C-0016/PIN.md`
- `canon/witnesses/C-0016/verify.py`
- `scratch/proof-system--c0016-rosati-purity-loewner-audit/AUDIT.md`
- `canon/claims/C-0016-function-field-polarization-positive-loewner-kernel.md`

7 Scope

The all-order kernel theorem applies to smooth projective geometrically connected curves over finite fields. The spectral Gram statement holds at every positive node, and the closed-point Euler series is used for $\operatorname{Re} z > 1/2$.

8 Sources

- Canonical claim: `canon/claims/C-0016-function-field-polarization-positive-loewner-kernel.md`
- Witness pin: `canon/witnesses/C-0016/PIN.md`
- Source and proof audit: `scratch/proof-system--c0016-rosati-purity-loewner-audit/AUDIT.md`